

B.E.2nd Sem. Examination May, 2008
Paper : Math-102-E

Time Allowed : 3 Hrs.

Maximum Marks : 100

Note:- Attempt **five** questions in all, selecting at least **one** question from each section.

SECTION-A

1. (a) Find the rank of $\begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$ by reducing it to the normal form. 10

Solution. Let $A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$

operating $R_3 \rightarrow R_3 - (R_1 + R_2)$

$$A \sim \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 2 & 3 & 5 & 4 \end{bmatrix}$$

operating $R_3 \rightarrow R_3 - R_2, R_2 \rightarrow R_2 - R_1$

$$A \sim \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

operating $C_1 \rightarrow \frac{1}{2}C_1, C_3 \rightarrow C_3 - 2C_2, C_4 \rightarrow \frac{1}{4}C_4$

$$\sim \begin{bmatrix} 0 & 2 & -1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

operating $C_2 \rightarrow C_2 - C_1, C_4 \rightarrow C_4 + C_3$

$$\sim \begin{bmatrix} 0 & 2 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

operating $C_2 \rightarrow C_2 + 2C_3, C_3 \rightarrow -C_3$

$$\sim \begin{bmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

operating $C_1 \leftrightarrow C_3$

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$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Operating $C_2 \leftrightarrow C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} I_2 & : & 0 \\ \\ 0 & : & 0 \end{bmatrix}$$

Which is required normal form. $\therefore \rho(A) = 2$. **Ans.**

1. (b) Find the inverse of : $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ using elementary transformations. 10

Solution. Writing the given matrix side by side with unit matrix I_3 , we get

$$[A:I_3] = \begin{bmatrix} 2 & 0 & -1 & : & 1 & 0 & 0 \\ 5 & 1 & 0 & : & 0 & 1 & 0 \\ 0 & 1 & 3 & : & 0 & 0 & 1 \end{bmatrix}$$

operating $R_2 \rightarrow R_2 - \frac{5}{2}R_1$

$$\sim \begin{bmatrix} 2 & 0 & -1 & : & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & : & -\frac{5}{2} & 1 & 0 \\ 0 & 1 & 3 & : & 0 & 0 & 1 \end{bmatrix}$$

operating $R_3 \rightarrow R_3 - R_2$

$$\sim \begin{bmatrix} 2 & 0 & -1 & : & 1 & 0 & 0 \\ 5 & 1 & \frac{5}{2} & : & -\frac{5}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{2} & : & \frac{5}{2} & -1 & 1 \end{bmatrix}$$

operating $R_1 \rightarrow R_1 + 2R_2$ & $R_2 \rightarrow R_2 - 5R_3$

$$\sim \begin{bmatrix} 2 & 0 & 0 & : & 6 & -2 & 2 \\ 0 & 1 & 0 & : & -15 & 6 & -5 \\ 0 & 0 & \frac{1}{2} & : & \frac{5}{2} & -1 & 1 \end{bmatrix}$$

operating $R_1 \rightarrow \frac{1}{2}R_1$ & $R_3 \rightarrow 2R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 3 & -1 & 1 \\ 0 & 1 & 0 & : & -15 & 6 & -5 \\ 0 & 0 & 1 & : & 5 & -2 & 2 \end{bmatrix} = [I_3 : A^{-1}]$$

$$\therefore A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} \text{ **Ans.**}$$

2. (a) Solve the equation by matrix method :

$$x + y + z = 6, \quad x - y + 2z = 5, \quad 3x + y + z = 8$$

10

Solution. In Matrix notation, the given system of equations can be written as $AX = B$

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where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 6 \\ 5 \\ 8 \end{bmatrix}$$

$$\therefore \text{Augmented Matrix } [A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & -1 & 2 & : & 5 \\ 3 & 1 & 1 & : & 8 \end{bmatrix}$$

Operating $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 1 & : & -1 \\ 0 & -2 & -2 & : & -10 \end{bmatrix}$$

operating $R_2 \rightarrow \frac{-1}{2}R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & -\frac{1}{2} & : & \frac{1}{2} \\ 0 & -2 & -2 & : & -10 \end{bmatrix}$$

operating $R_1 \rightarrow R_1 - R_2, R_3 \rightarrow R_3 + 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & \frac{3}{2} & : & \frac{11}{2} \\ 0 & 1 & -\frac{1}{2} & : & \frac{1}{2} \\ 0 & 0 & -3 & : & -9 \end{bmatrix}$$

operating $R_3 \rightarrow -\frac{1}{3}R_3$

$$\sim \begin{bmatrix} 1 & 0 & \frac{3}{2} & : & \frac{11}{2} \\ 0 & 1 & -\frac{1}{2} & : & \frac{1}{2} \\ 0 & 0 & -1 & : & 3 \end{bmatrix}$$

operating $R_1 \rightarrow R_1 - \frac{3}{2}R_3, R_2 \rightarrow R_2 + \frac{1}{2}R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$$

Hence $x = 1, y = 2, z = 3$. **Ans.**

2. (b) Verify Cayley-Hamilton theorem for the matrix :

$$A = \begin{bmatrix} 2 & 6 & 1 \\ 0 & 1 & -6 \\ 3 & 4 & -2 \end{bmatrix} \text{ and hence find } A^{-1}.$$

Solution. The characteristic equation of A is

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$$|A - \lambda I| = 0 \quad \text{i.e.,} \quad \begin{vmatrix} 2-\lambda & 6 & 1 \\ 0 & 1-\lambda & -6 \\ 3 & 4 & -2-\lambda \end{vmatrix} = 0$$

or $\lambda^3 - \lambda^2 + 17\lambda + 67 = 0$ (on simplification)

To verify Cayley - Hamilton theorem, we have to show that

$$A^3 - A^2 + 17A + 67I = 0 \quad \dots\dots(i)$$

Now

$$A^2 = \begin{bmatrix} 2 & 6 & 1 \\ 0 & 1 & -6 \\ 3 & 4 & -2 \end{bmatrix} \times \begin{bmatrix} 2 & 6 & 1 \\ 0 & 1 & -6 \\ 3 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 7 & 22 & -36 \\ -18 & -23 & 6 \\ 0 & 14 & -17 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 7 & 22 & -36 \\ -18 & -23 & 6 \\ 0 & 14 & -17 \end{bmatrix} \times \begin{bmatrix} 2 & 6 & 1 \\ 0 & 1 & -6 \\ 3 & 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -94 & -80 & -53 \\ -18 & -107 & 108 \\ -51 & -54 & -50 \end{bmatrix}$$

$$\therefore A^3 - A^2 + 17A + 67I$$

$$= \begin{bmatrix} -94 & -80 & -53 \\ -18 & -107 & 108 \\ -51 & -54 & -50 \end{bmatrix} - \begin{bmatrix} 7 & 22 & -36 \\ -18 & -23 & 6 \\ 0 & 14 & -17 \end{bmatrix} + 17 \begin{bmatrix} 2 & 6 & 1 \\ 0 & 1 & -6 \\ 3 & 4 & -2 \end{bmatrix} + 67 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

This verifies Cayley-Hamilton theorem .

Now, Multiplying both sides of (i) by A^{-1} , we have

$$A^2 - A + 17I + 67A^{-1} = 0$$

$$\Rightarrow 67A^{-1} = -A^2 + A - 17I$$

$$\Rightarrow 67A^{-1} = \begin{bmatrix} -7 & -22 & 36 \\ 18 & 23 & -6 \\ 0 & -14 & 17 \end{bmatrix} + \begin{bmatrix} 2 & 6 & 1 \\ 0 & 1 & -6 \\ 3 & 4 & -2 \end{bmatrix} - \begin{bmatrix} 17 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 17 \end{bmatrix}$$

$$= \begin{bmatrix} -22 & -16 & 37 \\ 18 & 7 & -12 \\ 0 & -10 & -2 \end{bmatrix} \quad \therefore A^{-1} = \frac{1}{67} \begin{bmatrix} -22 & -16 & 37 \\ 18 & 7 & -12 \\ 3 & -10 & -2 \end{bmatrix} \quad \text{Ans.}$$

SECTION - B

3. (a) Solve the D.E. $(2xy + y - \tan y)dx + (x^2 - x \tan^2 y + \sec^2 y)dy = 0$.

Solution. Here $M = 2xy + y - \tan y$ and $N = x^2 - x \tan^2 y + \sec^2 y$

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$$\therefore \frac{\partial M}{\partial y} = 2x + 1 - \sec^2 y, \quad \therefore \frac{\partial N}{\partial y} = 2x - \tan^2 y = 2x + 1 - \sec^2 y$$

Here $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Thus the given eq. is exact and its solution is

$$\int_{y \text{ const.}} M dx + \int (\text{terms of } N \text{ not containing } x) dy = C$$

$$\int_{y \text{ const.}} 2xy + y - \tan y dx + \int \sec^2 y dy = C$$

$$\frac{2x^2}{2} y + xy - x \tan y + \tan y = C$$

$$x^2 y + xy - x \tan y + \tan y = c \quad \text{Ans.}$$

3. (b) Solve the D.E. : $\left(y + \frac{y^3}{3} + \frac{x^2}{2}\right) dx + \frac{1}{4}(x + xy^2) dy = 0$ 10

Solution. The given equation is $\left[y + \frac{y^3}{3} + \frac{x^2}{2}\right] dx + \frac{1}{4}(x + xy^2) dy = 0$ (i)

Comparing (i) with $M dx + N dy = 0$, we get

$$M = y + \frac{y^3}{3} + \frac{x^2}{2} \quad \text{and} \quad N = \frac{1}{4}(x + xy^2)$$

$$\therefore \frac{\partial M}{\partial y} = 1 + y^2 \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{1}{4}(1 + y^2)$$

since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, thus equation (i) is not exact.

Now $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{(1 + y^2) - \frac{1}{4}(1 + y^2)}{\frac{1}{4}x(1 + y^2)} = \frac{\{4(1 + y^2) - (1 + y^2)\} / 4}{\frac{1}{4}x(1 + y^2)} = \frac{3(1 + y^2)}{x(1 + y^2)} = \frac{3}{x}$

which is a function of x only

$$\therefore \text{I.F.} = e^{\int \frac{1}{x} dx} = e^{\log x} = e^{\log x^3} = x^3$$

Multiplying eq. (i) by x^3 , we get

$$\left(yx^3 + \frac{y^3}{3}x^3 + \frac{x^5}{2}\right) dx + \frac{1}{4}x^4(1 + y^2) dy = 0$$

Which is exact . $\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = x^3(1 + y^2)$

Hence solution is $\int_{y \text{ const.}} \left[yx^3 + \frac{x^3 y^3}{3} + \frac{x^5}{2}\right] dx = C_1$

$$\frac{y x^4}{4} + \frac{x^4 y^3}{12} + \frac{x^6}{12} = C_1$$

$$\frac{3yx^4 + x^4 y^3 + x^6}{12} = C_1$$

or

$$3y x^4 + x^4 y^3 + x^6 = 12 c_1 = C. \text{ Ans.}$$

4. (a) The charge Q on the plate of a condenser of capacity C charged through a resistance R by a steady voltage V satisfies the differential equation.

$$R \frac{dQ}{dt} + \frac{Q}{C} = V \quad \text{If } Q = 0 \text{ at } t = 0 \text{ show that : } Q = CV \left(1 - e^{-\frac{t}{RC}} \right) \quad 10$$

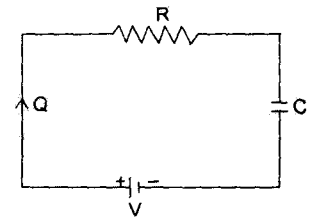
Solution. According to question

$$R \frac{dQ}{dt} + \frac{Q}{C} = V \quad \text{or} \quad \frac{dQ}{dt} + \frac{Q}{CR} = \frac{V}{R} \quad \dots(i)$$

$$\therefore \text{I.F.} = e^{\int \frac{1}{CR} dt} = e^{\frac{t}{CR}}$$

The Solution of Eq. (i) is $e^{\frac{t}{CR}} \cdot Q = \int \frac{V}{R} e^{\frac{t}{CR}} dt + k$

$$= \frac{V}{R} \frac{e^{\frac{t}{CR}}}{\frac{1}{CR}} + K \quad \text{or} \quad Q = VC + Ke^{-\frac{t}{CR}} \quad \dots(ii)$$



But $Q=0$ at $t=0$, $0 = VC + k$ or $k = -VC$

From (ii) $Q = VC - VCe^{-\frac{t}{CR}} = VC \left[1 - e^{-\frac{t}{CR}} \right]$. **proved**

4. (b) Solve the D.E. : $(D-2)^2 y = 8(e^{2x} + \sin 2x + x^2)$. 10

Solution. A.E. is $(D-2)^2 = 0$ whence $D = 2, 2$

$$\therefore \text{C.F.} = (c_1 + c_2 x)e^{2x}$$

$$\text{P.I.} = \frac{1}{(D-2)^2} [8(e^{2x} + \sin 2x + x^2)]$$

$$= 8 \left[\frac{1}{(D-2)^2} e^{2x} + \frac{1}{(D-2)^2} \sin 2x + \frac{1}{(D-2)^2} x^2 \right]$$

Now $\frac{1}{(D-2)^2} e^{2x} = x \cdot \frac{1}{2(D-2)} e^{2x}$ [Case of failure]

$$= x^2 \cdot \frac{1}{2} e^{2x}$$
 [Case of failure]

$$= \frac{x^2}{2} e^{2x}$$

$$\frac{1}{(D-2)^2} \sin 2x = \frac{1}{D^2 - 4D + 4} \sin 2x = \frac{1}{-2^2 - 4D + 4} \sin 2x \quad [\text{Putting } D^2 = -2^2]$$

$$-\frac{1}{4D} \sin 2x = -\frac{1}{4} \int \sin 2x \, dx = -\frac{1}{4} \left(-\frac{\cos 2x}{2} \right) = \frac{1}{8} \cos 2x$$

$$\frac{1}{(D-2)^2} x^2 = \frac{1}{(2-D)^2} x^2 = \frac{1}{4 \left(1 - \frac{D}{2} \right)^2} x^2 = \frac{1}{4} \left(1 - \frac{D}{2} \right)^{-2} x^2$$

$$= \frac{1}{4} \left[1 - 2 \left(-\frac{D}{2} \right) + \frac{(-2)(-3)}{2} \left(\frac{D}{2} \right)^2 \dots \right] x^2$$

$$= \frac{1}{4} \left[1 + D + \frac{3}{4} D^2 + \dots \right] x^2 = \frac{1}{4} \left[x^2 + D(x^2) + \frac{3}{4} D^2(x^2) \right]$$

$$\begin{aligned} \text{P.I.} &= 8 \left[\frac{x^2}{2} e^{2x} + \frac{1}{8} \cos 2x + \frac{1}{4} \left(x^2 + 2x + \frac{3}{2} \right) \right] \\ &= 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3 \end{aligned}$$

Hence the C.S. is $y = (c_1 + c_2 x) e^{2x} + 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3$. **Ans.**

5. (a) Solved by method of variation of parameters : $\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} = e^x \sin x$ 10

Solution. Given equation in symbolic form is $(D^2 - 2D)y = e^x \sin x$

ts A.E. is $D^2 - 2D = 0$ so that $D = 0, 2$

\therefore C.F. is $y = c_1 e^0 + c_2 e^{2x} = c_1 + c_2 e^{2x}$

Here $y_1 = 1, y_2 = e^{2x}$ and $X = e^x \sin x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix} = 2e^{2x}$$

$$\text{P.I.} = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$

$$= - \int \frac{e^{2x} \times e^x \sin x}{2e^{2x}} dx + e^{2x} \int \frac{e^x \sin x}{2e^{2x}} dx$$

$$= -\frac{1}{2} \int e^x \sin x \, dx + \frac{e^{2x}}{2} \int e^{-x} \sin x \, dx$$

$$= -\frac{1}{2} \left[\frac{e^x}{1+1} (\sin x - \cos x) \right] + \frac{e^{2x}}{2} \left[\frac{e^{-x}}{1+1} (-\sin x - \cos x) \right]$$

$$= -\frac{1}{4} e^x (\sin x - \cos x) + \frac{e^x}{4} (-\sin x - \cos x)$$

$$= \frac{e^x}{4} [-\sin x + \cos x - \sin x - \cos x] = \frac{-2e^x}{4} \sin x = -\frac{e^x}{2} \sin x$$

Hence the C.S. is $y = c_1 + c_2 e^{2x} - \frac{e^x}{2} \sin x$. **Ans.**

5. (b) Solve :

$$\frac{dx}{dt} + 4x + 3y = t$$

$$\frac{dy}{dt} + 2x + 5y = e^t$$

10

Solution. Write D for $\frac{d}{dt}$, the given equation become $(D + 4)x + 3y = t$... (i)

and $2x + (D + 5)y = e^t$... (ii)

To eliminate y, operating on both sides of (i) by $(D + 5)$ and on both sides of (ii) by 3 and subtracting

we get $[(D + 4)(D + 5) - 6]x = (D + 5)t - 3e^t$

or $(D^2 + 9D + 14)x = 1 + 5t - 3e^t$

Its A. E. is $D^2 + 9D + 14 = 0$

or $(D + 2)(D + 7) = 0 \quad \therefore D = -2, -7$

$$C.F. = c_1 e^{-2t} + c_2 e^{-7t}$$

$$P.I. = \frac{1}{D^2 + 9D + 14} (1 + 5t - 3e^t)$$

$$= \frac{1}{D^2 + 9D + 14} e^{0t} + 5 \frac{1}{D^2 + 9D + 14} t - 3 \frac{1}{D^2 + 9D + 14} e^t$$

$$= \frac{1}{0^2 + 9(0) + 14} e^{0t} + 5 \frac{1}{14 \left(1 + \frac{9D}{14} + \frac{D^2}{14} \right)} t - 3 \frac{1}{1^2 + 9(1) + 14} e^t$$

$$= \frac{1}{14} + \frac{5}{14} \left[1 + \left(\frac{9D}{14} + \frac{D^2}{14} \right) \right]^{-1} t - \frac{1}{8} e^t = \frac{1}{14} + \frac{5}{14} \left[1 - \left(\frac{9D}{14} + \frac{D^2}{14} \right) + \dots \right] t - \frac{1}{8} e^t$$

$$= \frac{1}{14} + \frac{5}{14} \left(t - \frac{9}{14} \right) - \frac{1}{8} e^t = \frac{1}{14} + \frac{5}{14} t - \frac{45}{196} - \frac{1}{8} e^t = \frac{5}{14} t - \frac{31}{196} - \frac{1}{8} e^t$$

$$\therefore x = c_1 e^{-2t} + c_2 e^{-7t} + \frac{5}{14} t - \frac{31}{196} - \frac{1}{8} e^t$$

Now $\frac{dx}{dt} = -2c_1 e^{-2t} - 7c_2 e^{-7t} + \frac{5}{14} - \frac{1}{8} e^t$

Substituting the values of x and $\frac{dx}{dt}$ in (i), we have $3y = t - \frac{dx}{dt} - 4x$

$$= t + 2c_1 e^{-2t} + 7c_2 e^{-7t} - \frac{5}{14} + \frac{1}{8} e^t - 4c_1 e^{-2t} - 4c_2 e^{-7t} - \frac{10}{7} t + \frac{31}{49} + \frac{1}{2} e^t$$

$$\therefore y = \frac{1}{3} \left[-2c_1 e^{-2t} + 3c_2 e^{-7t} - \frac{3}{7} t + \frac{27}{98} + \frac{5}{8} e^t \right]$$

Hence $x = c_1 e^{-2t} + c_2 e^{-7t} + \frac{5}{14} t - \frac{31}{196} - \frac{1}{8} e^t$

$$y = -\frac{2}{3} c_1 e^{-2t} + c_2 e^{-7t} - \frac{1}{7} t + \frac{9}{98} + \frac{5}{24} e^t. \text{ Ans.}$$

SECTION - C

6. (a) Find the Laplace transform of : (i) $t \sin^3 t$ (ii) $e^{-t} \frac{\sin t}{t}$

10

Solution. (i)

$$\begin{aligned}
 & L[t \sin^3 t] \\
 &= L\left[\frac{t}{4}(3 \sin t - \sin 3t)\right] \quad [\because \sin 3A = 3 \sin A - 4 \sin^3 A] \\
 &= \frac{1}{4}[3L(t \sin t) - L(t \sin 3t)] \\
 &= \frac{1}{4}\left[3 \times \frac{-d}{ds}\left[\frac{1}{s^2+1}\right] + \frac{d}{ds}\left[\frac{3}{s^2+9}\right]\right] \\
 &= \frac{1}{4}\left[-3\left\{\frac{(s^2+1) \times 0 - 1 \times 2s}{(s^2+1)^2}\right\} + 3\left\{\frac{(s^2+9) \times 0 - 2s}{(s^2+9)^2}\right\}\right] \\
 &= \frac{3}{4}\left[\frac{2s}{(s^2+1)^2} - \frac{2s}{(s^2+9)^2}\right] \\
 &= \frac{3s}{2}\left[\frac{1}{(s^2+1)^2} - \frac{1}{(s^2+9)^2}\right] \\
 &= \frac{3s}{2}\left[\frac{s^4+81+18s^2-s^4-1-2s^2}{(s^2+1)^2(s^2+9)^2}\right] \\
 &= \frac{3s}{2}\left[\frac{16s^2+80}{(s^2+1)^2(s^2+9)^2}\right] \\
 &= \frac{3s(8s^2+40)}{(s^2+1)^2(s^2+9)^2} = \frac{24s(s^2+5)}{(s^2+1)^2(s^2+9)^2} \cdot \text{Ans.}
 \end{aligned}$$

Solution. (ii)

$$\begin{aligned}
 & L(e^{-t} \sin t) = \frac{1}{(s+1)^2+1} \\
 \Rightarrow & L\left(\frac{e^{-t} \sin t}{t}\right) = \int_s^\infty \frac{1}{(s+1)^2+1} ds = \left[\tan^{-1}(s+1)\right]_s^\infty \\
 &= \frac{\pi}{2} - \tan^{-1}(s+1) = \cot^{-1}(s+1) \cdot \text{Ans.}
 \end{aligned}$$

6. (b) Find the inverse Laplace transform of : $\frac{s^2}{(s+1)^3}$

10

Solution. Let $\frac{s^2}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$

Multiplying by $(s+1)^3$, we have

$$s^2 = A(s+1)^2 + (s+1) + C$$

$$s^2 = A(s^2 + 2s + 1) + Bs + B + C$$

$$s^2 = (A)s^2 + (2A + B)s + A + B + C \quad \dots(i)$$

$$\text{Equating Coefficients of } s^2, A = 1 \quad \dots(ii)$$

$$\text{Equating Coefficients of } s, 2A + B = 0 \quad \dots(iii)$$

$$\text{Equating constant terms } A + B + C = 0 \quad \dots(iv)$$

$$\text{from (ii) and (iii)} \quad 2 + B = 0 \Rightarrow B = -2$$

$$\text{from (iv)} \quad 1 - 2 + C = 0 \Rightarrow C = 1$$

$$\therefore L^{-1} \left[\frac{s^2}{(s+1)^3} \right] = L^{-1} \left[\frac{1}{s+1} \right] - 2L^{-1} \left[\frac{1}{(s+1)^2} \right] + L^{-1} \left[\frac{1}{(s+1)^3} \right]$$

$$= e^{-t} - 2e^{-t} \times \frac{t}{1!} + e^{-t} \times \frac{t^2}{2!} = e^{-t} \left[1 - 2t + \frac{t^2}{2} \right]. \text{ Ans.}$$

7. (a) Find the Laplace transform of function $f(t)$ given by :

$$f(t) = \begin{cases} t & 0 < t < c \\ 2c - t & c < t < 2c \end{cases}$$

10

$$\begin{aligned} \text{Solution. } L\{f(t)\} &= \frac{1}{1 - e^{-2cs}} \int_0^{2c} e^{-st} f(t) dt = \frac{1}{1 - e^{-2cs}} \left[\int_0^c e^{-st} \cdot t dt + \int_c^{2c} e^{-st} (2c - t) dt \right] \\ &= \frac{1}{1 - e^{-2cs}} \left[\left\{ t \cdot \frac{e^{-st}}{-s} - 1 \cdot \frac{e^{-st}}{s^2} \right\}_0^c + \left\{ (2c - t) \cdot \frac{e^{-st}}{-s} - (-1) \cdot \frac{e^{-st}}{s^2} \right\}_c^{2c} \right] \\ &= \frac{1}{1 - e^{-2cs}} \left[\left\{ -\frac{ce^{-cs}}{s} - \frac{e^{-cs}}{s^2} + \frac{1}{s^2} \right\} + \left\{ \frac{e^{-2cs}}{s^2} + \frac{ce^{-cs}}{s} - \frac{e^{-cs}}{s^2} \right\} \right] \\ &= \frac{1}{1 - e^{-2cs}} \left(\frac{1 - 2e^{-cs} + e^{-2cs}}{s^2} \right) = \frac{1}{s^2} \cdot \frac{(1 - e^{-cs})^2}{(1 + e^{-cs})(1 - e^{-cs})} = \frac{1}{s^2} \cdot \frac{1 - e^{-cs}}{1 + e^{-cs}} \\ &= \frac{1}{s^2} \left(\frac{e^{cs/2} - e^{-cs/2}}{e^{cs/2} + e^{-cs/2}} \right) = \frac{1}{s^2} \tanh \frac{cs}{2}. \text{ Ans.} \end{aligned}$$

7.(b) Form the PDE by eliminating the arbitrary functions from : $z = x f(x+t) + g(x+t)$. 10

Solution. Given $z = (xf(x+t) + g(x+t))$

$$\therefore \frac{\partial z}{\partial x} = f(x+t) + xf'(x+t) + g'(x+t)$$

$$\frac{\partial^2 z}{\partial^2 x} = f'(x+t) + f'(x+t) + xf''(x+t) + g''(x+t)$$

$$\text{or } \frac{\partial^2 z}{\partial x^2} = 2f'(x+t) + xf''(x+t) + g''(x+t) \quad \dots(i)$$

$$\frac{\partial z}{\partial t} = xf'(x+t) + g'(x+t)$$

$$\frac{\partial^2 z}{\partial t^2} = xf''(x+t) + g''(x+t) \quad \dots(ii)$$

Subtracting (ii) from (i) $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial t^2} = 2f'(x+t) \quad \dots(iii)$

Also
$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial t} &= f'(x+t) + xf''(x+t) + g''(x+t) \\ &= \frac{1}{2} \left[\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial t^2} \right] + \frac{\partial^2 z}{\partial t^2} \end{aligned} \quad \text{[using (ii) and (iii)]}$$

or
$$\frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial^2 z}{\partial x \partial t} + \frac{\partial^2 z}{\partial t^2} = 0$$

Which is a partial differential equation of the second order. **Ans.**

8. (a) Solve : $z^2 (p^2 + q^2) = 1$

10

Solution. The given equation can be written as

$$z^2 \left[\left(x \frac{\partial z}{\partial X} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right] = 1 \quad \dots(i)$$

Put $\frac{dx}{x} = dX$ so that $X = \log x$

and

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{1}{x} \quad \text{i.e. } x \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}$$

\therefore Equation (i) reduces to $z^2 \left[\left(\frac{\partial z}{\partial X} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right] = 1$

or $z^2 (P^2 + q^2) = 1$, where $P = \frac{\partial z}{\partial X} \quad \dots(ii)$

Let $u = X + ay$ so that $P = \frac{dz}{du}$ and $q = a \frac{dz}{du}$

\therefore From (ii) we have $z^2 \left[\left(\frac{dz}{du} \right)^2 + a^2 \left(\frac{dz}{du} \right)^2 \right] = 1$

$$(1+a^2)z^2 \left(\frac{dz}{du} \right)^2 = 1 \quad \text{or} \quad \sqrt{1+a^2} \cdot z \, dz = \pm du$$

Integrating,
$$\sqrt{1+a^2} \cdot \frac{z^2}{2} = \pm u + b$$

or
$$\sqrt{1+a^2} \cdot z^2 = \pm 2(X + ay) + 2b$$

or
$$\sqrt{1+a^2} \cdot z^2 = \pm 2(\log x + ay) + c$$

Which is the required complete solution. **Ans.**

8. (b) Determine the solution of one-dimensional heat equation : $\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$
where $u(0, t) = 0 = u(l, t)$ ($t > 0$) and $u(x, 0) = x$, l being length of the bar. 10

Solution. The temperature function $u(x, t)$ satisfies the differential equation $\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$

\therefore Solution of the heat equation is $u(x, t) = (c_1 \cos px + c_2 \sin px)e^{-c^2 p^2 t}$ (i)

But $u(0, t) = 0$, from (i)

$$0 = c_1 e^{-c^2 p^2 t} \Rightarrow c_1 = 0$$

\Rightarrow From (i), $u(x, t) = c_2 \sin px \cdot e^{-c^2 p^2 t}$ (ii)

since $u(l, t) = 0$, from (ii)

$$0 = c_2 \sin pl e^{-c^2 p^2 t}$$

$$\sin pl = 0 \Rightarrow pl = n\pi$$

$$p = \frac{n\pi}{l}, \text{ n bearing an integer}$$

\therefore solution of (ii) is $u(x, t) = bn \sin\left(\frac{n\pi x}{l}\right) e^{\frac{-c^2 n^2 \pi^2 t}{l^2}}$ on replacing c_2 by bn .

The most general solution is obtained by adding all such solutions for $n=1, 2, 3, \dots$

$$\therefore u(x, t) = \sum_{n=1}^{\infty} bn \sin\left(\frac{n\pi x}{l}\right) e^{\frac{-c^2 n^2 \pi^2 t}{l^2}} \quad \dots\text{... (iii)}$$

since $u(x, 0) = f(x)$, we have $f(x) = \sum_{n=1}^{\infty} bn \sin\left(\frac{n\pi x}{l}\right)$

By Fourier series $bn = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

But $f(x) = x$ $\therefore bn = \frac{2}{l} \int_0^l x \sin\left(\frac{n\pi x}{l}\right) dx$

$$= \frac{2}{l} \left[\left\{ -x \frac{1}{n\pi} \cos \frac{n\pi x}{l} \right\}_0^l + \frac{1}{n\pi} \int_0^l \cos \frac{n\pi x}{l} dx \right]$$

$$= \frac{2}{l} \left[\frac{-l^2}{n\pi} \cos n\pi + \frac{1}{n\pi} \cdot \frac{1}{n\pi} \left(\sin \frac{n\pi x}{l} \right)_0^l \right] = \frac{-2l}{n\pi} \cos n\pi$$

$$\therefore u(x, t) = \sum_{n=1}^{\infty} \frac{-2l}{n\pi} \cos n\pi \sin\left(\frac{n\pi x}{l}\right) e^{\frac{-c^2 n^2 \pi^2 t}{l^2}}$$

$$= \frac{-2l}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\pi}{n} \sin\left(\frac{n\pi x}{l}\right) e^{\frac{-c^2 n^2 \pi^2 t}{l^2}}$$