B.Tech.

First Semester Examination, 2010-2011

Mathematics-I (Math-101-F)

Note : Attempt five questions in total, selecting one question from each unit. Question number 1 is compulsory. All questions carry equal marks.

Q. 1. (a) Discuss the convergence and divergence of the geometric series.

Ans. Let us consider the geometric series

$$a + ax + ax^2 + ax^3 + \ldots\ldots\ldots\ldots + ax^{n-1} + \ldots\ldots \quad \ldots(1)$$

Let S_n be the sum of first n terms of equation (1).

$$S_n = \frac{a\left(1-x^n\right)}{1-x} \text{ if } x < 1$$

And
$$S_n = \frac{a\left(x^n-1\right)}{x-1} \text{ if } x > 1.$$

Case 1 : When $|x| < 1$

i.e., $-1 < x < 1$

If $|x| < 1$, then $x^n \to 0$ as $n \to \infty$

$$\therefore \qquad \lim S_n = \lim \frac{a\left(1-x^n\right)}{1-x} = \frac{a(1-0)}{1-x} = \frac{a}{1-x}$$

Which is definite a finite number and therefore the series is convergent.

Case 2 : When $x = 1$

Then each term of series (1) is a

i.e., $S_n = a + a + \ldots\ldots + $ up to n terms

$$S_n = na$$

$\lim S_n = \infty$ or $-\infty$ according as a is positive or negative, hence the series is divergent.

Case 3 : When $x > 1$, then $x^n \to \infty$ as $n \to \infty$

$$\therefore \qquad \lim S_n = \lim \frac{a\left(x^n-1\right)}{(x-1)} = \infty \text{ or } -\infty \text{ as } a > 0 \text{ or } a < 0.$$

Hence the series is divergent.

Case 4 : When $x = -1$, then the series (1) becomes

$$S = a - a + a - \ldots + \ldots\ldots$$

The sum of n terms of the series is a or 0 according as n is odd or even. Hence, the series is an oscillatory series, the oscillation being infinite.

Case 5 : When $x < -1$

If. $x < -1$, then $-x > 1$

Let $r = -x$, then $r > 1$

And $r^n \to \infty$ as $n \to \infty$

Now,
$$S_n = \frac{a(1-x^n)}{1-x} = \frac{a\{1-(-r)^n\}}{1-(-r)}$$

$$= \frac{a(1+r^n)}{1+r} \text{ or } \frac{a(1-r^n)}{(1+r)}$$

According as n is odd or even.

$\lim S_n = \infty$ or $-\infty$ according as $a > 0$ and if $a < 0$, the results are reversed. Therefore, in this case the series is an oscillatory series, the oscillation being infinite.

Hence, a geometric series when common ratio is x is convergent if $|x| < 1$, divergent if $x \geq 1$ and oscillatory if $x \leq -1$.

Q. 1. (b) Test for convergence the series whose nth term is : $\dfrac{n^{n^2}}{(n+1)^{n^2}}$.

Ans. Given : the nth term is,

$$u_n = \frac{n^{n^2}}{(n+1)^{n^2}}$$

$$u_n = \left(\frac{n}{n+1}\right)^{n^2}$$

$$u_n^{1/n} = \left(\frac{n}{n+1}\right)^n$$

$$u_n^{1/n} = \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$= \frac{1}{e} \text{ which is } < 1$$

By Cauchy Root Test, the given series is convergent.

Q. 1. (c) Define rank of a matrix. Determine rank of matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}.$$

Ans. Rank of a Matrix : The rank of a matrix is said to be r if—

It has at least one non-zero minor of order. Every minor of A of order higher than r is zero.

Now, the given matrix is,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

$$R_2 \to R_2 - R_1,\ R_3 \to R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_2 \to C_2 - 2C_1,\ C_3 \to C_3 - 3C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_3 \to C_3 + C_2/2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_2/2$$

$$\sim\left[\begin{array}{cc|c}1 & 0 & 0\\ 0 & 1 & 0\\ \hline 0 & 0 & 0\end{array}\right]$$

$$\sim\left[\begin{array}{c|c}I_2 & 0\\ 0 & 0\end{array}\right]$$

Hence, rank of the given matrix is 2.

Q. 1. (d) Using Cayley-Hamilton theorem, find the inverse of matrix :

$$\begin{bmatrix}5 & 3\\ 3 & 2\end{bmatrix}.$$

Ans. Given matrix is,

$$[A] = \begin{bmatrix}5 & 3\\ 3 & 2\end{bmatrix}$$

The characteristic equation is,

$$|A - \lambda I| = 0$$

$$[A - I\lambda] = \begin{bmatrix}5 & 3\\ 3 & 2\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$[A - I\lambda] = \begin{bmatrix}5-\lambda & 3\\ 3 & 2-\lambda\end{bmatrix}$$

$$[A - I\lambda] = \begin{vmatrix}5-\lambda & 3\\ 3 & 2-\lambda\end{vmatrix} = 0$$

$$(5-\lambda)(2-\lambda) - 9 = 0$$

$$10 - 7\lambda + \lambda^2 - 9 = 0$$

$$\lambda^2 - 7\lambda + 1 = 0$$

Using Caylay Hamilton Theorem,

$$A^2 - 7A + I = 0$$

Multiplying both sides by A,

$$A - 7A^{-1}A + A^{-1} = 0$$

$$A^{-1} = -A + 7I$$

$$= -\begin{bmatrix}5 & 3\\ 3 & 2\end{bmatrix} + \begin{bmatrix}7 & 0\\ 0 & 7\end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \qquad \textbf{Ans.}$$

Q. 1. (e) Find the asymptotes of the curve :

$$x^3 + y^3 - 3axy = 0.$$

Ans. Given curve is $x^3 + y^3 - 3axy = 0$.

(i) Asymptote Parallel to x-axis : No asymptote parallel to x-axis because the coefficient of highest power of x is constant.

(ii) Asymptote Parallel to y-axis : Since the coefficient of highest power of y is constant, there is no asymptote parallel to y-axis.

(iii) Oblique Asymptote : Putting $y = mx + c$ in the equation,

$$x^3 + (mx + c)^3 - 3ax(mx + c) = 0$$

$$\Rightarrow \quad x^3\left(1 + m^3\right) + 3x^2\left(m^2c - am\right) + 3x(cm - ca) + c^3 = 0$$

Hence, $\quad 1 + m^3 = 0$

And $\quad m^2c - am = 0$

On solving,

$$m = -1, \; c = -a$$

Oblique asymptote is,

$$y = -x - a$$

Or $\quad \boxed{y + x + a = 0} \qquad$ **Ans.**

Q. 1. (f) If $u = \sin^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{y}{x}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.

Ans. Given : $\quad u = \sin^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{y}{x}$

u is a homogeneous function of degree 0. Hence, by Euler's theorem.

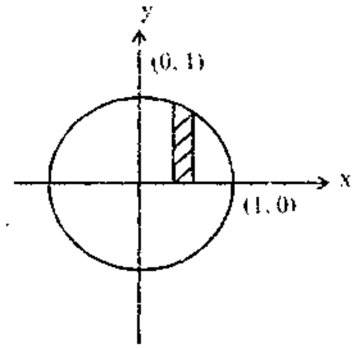
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Here, $n = 0$

Hence, $\quad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.

Q. 1. (g) Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}} y^2\,dx\,dy$ by changing the order of integration.

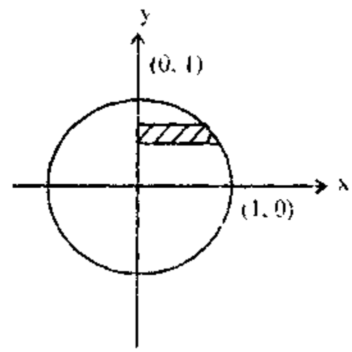
Ans. Let
$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, dy \, dx$$



$$y = 0 \text{ to } y = \sqrt{1-x^2}$$

$$x^2 + y^2 = 1$$

By changing the order of integration,



$$I = \int_0^1 y^2 \int_0^{\sqrt{1-y^2}} dx \, dy$$

$$= \int_0^1 y^2 [x]_0^{\sqrt{1-y^2}} \, dy$$

$$= \int_0^1 y^2 \sqrt{1-y^2} \, dy$$

Let
$$y = \sin\theta$$

$$dy = \cos\theta \, d\theta$$

θ varies from 0 to $\pi/2$

$$I = \int_0^{\pi/2} \sin^2\theta \cos\theta . \cos\theta \, d\theta$$

$$= \int_0^{\pi/2} (\sin\theta \cos\theta)^2 \, d\theta$$

$$= \frac{1}{4} \int_0^{\pi/2} \sin^2 2\theta \, d\theta$$

$$= \frac{1}{4} \int_0^{\pi/2} \left(\frac{1-\cos 4\theta}{2} \right) d\theta$$

$$= \frac{1}{8} \left[\theta - \frac{\sin 4\theta}{4} \right]_0^{\pi/2}$$

$$= \frac{1}{8} \left[\frac{\pi}{2} \right]$$

$$\boxed{I = \frac{\pi}{16}}$$

Ans.

Q. 1. (h) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dz \, dy \, dx$.

Ans. Let.
$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} xy\left[\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}} dy\,dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{\sqrt{1-x^2}} xy\left(1-x^2-y^2\right)dy\,dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{\sqrt{1-x^2}} x\left(y-x^2y-y^3\right)dy\,dx$$

$$= \frac{1}{2}\int_0^1 x\left[\left(1-x^2\right)y\frac{2}{2}-y\frac{4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{\left(1-x^2\right)^2}{2}-\frac{\left(1-x^2\right)^2}{4}\right]dx$$

$$= \frac{1}{8}\int_0^1 x\left(1-x^2\right)^2 dx$$

Let $\qquad 1-x^2 = t$

$-2xdx = dt$

$$= -\frac{1}{16}\int_1^0 t^2 dt$$

$$= \frac{1}{16}\int_0^1 t^2\,dt = \frac{1}{48}$$ **Ans.**

Unit-I

Q. 2. (a) Show that the series $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$.

Ans. The given series is,

$$\Sigma \frac{1}{n^p}$$

Case 1 : When $p > 1$, since the terms of the given series are all positive, then

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots\ldots\ldots=\frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)+\ldots\ldots\ldots\ldots+\ldots\ldots\ldots\ldots\ldots$$

Now, since $\quad p > 1$

$$3 > 2$$

$$3^p > 2^p$$

$$\Rightarrow \quad \frac{1}{3^p} < \frac{1}{2^p}$$

$$\frac{1}{2^p}+\frac{1}{3^p} < \frac{1}{2^p}+\frac{1}{2^p}$$

$$\frac{1}{2^p}+\frac{1}{3^p} < \frac{2}{2^p}$$

Similarly,

$$\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p} < \frac{4}{4^p} \text{ and so on.}$$

Thus, the given series term by term is given by

$$< \frac{1}{1^p}+\frac{2}{2^p}+\frac{4}{4^p}+\frac{8}{8^p}+\ldots..$$

But the series on the R.H.S. is a geometric series and is convergent since the common ratio is $\frac{2}{2^p}=\frac{1}{2^{p-1}}$ which is less than 1 as $p > 1$.

Case 2 : When $p = 1$, the given series is

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots..$$

$$=1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\ldots\ldots$$

Now, as, $3 < 4$

$$\frac{1}{3} > \frac{1}{4}$$

$$\frac{1}{3}+\frac{1}{4} > \frac{1}{4}+\frac{1}{4}$$

$$\frac{1}{3}+\frac{1}{4} > \frac{1}{2}$$

Similarly, $\quad \frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2}$ and so on

The given series becomes,

$$> 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+......$$

$$= 1+(n-1)\frac{1}{2}$$

$$= \frac{1}{2}(n+1), \text{ which tends to } \infty \text{ as } n \Rightarrow \infty.$$

Hence, series is divergent when $p = 1$.

Case 3 : Let $p < 1$, then

$$\frac{1}{n^p} > \frac{1}{n}, n = 2, 3, 4......$$

In this case, the given series is greater than the series term-wise as,

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+......$$

which is divergent.

Hence, if $p \leq 1$, series divergent and if $p > 1$, series convergent.

Q. 2. (b) Find whether the series :

$$x+\frac{2^2x^2}{2!}+\frac{3^3x^3}{3!}+\frac{4^4x^4}{4!}+........ \textbf{ converges or diverges.}$$

Ans. According to the given series,

$$u_n = \frac{n^n x^n}{n!}$$

$$u_{n+1} = \frac{(n+1)^{n+1}x^{n+1}}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{1}{x}.\frac{n^n}{n!}.\frac{(n+1)!}{(n+1)^{n+1}}$$

$$= \frac{1}{x}\frac{n^n(n+1)}{(n+1)^{n+1}}$$

$$= \frac{1}{x}\frac{n^n}{(n+1)^n}$$

$$= \frac{1}{x} \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}.\frac{1}{e} = \frac{1}{ex}$$

By ratio test. Σu_n is convergent if $\frac{1}{ex} > 1$ i.e., if $x < \frac{1}{e}$.

and Σu_n is divergent if $\frac{1}{ex} < 1$ i.e., if $x > \frac{1}{e}$

For $$\frac{1}{ex} = 1$$

i.e., $x = \frac{1}{e}$, ratio test fails.

Applying log test

At $$x = 1$$

$$\frac{u_n}{u_{n+1}} = e.\frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\therefore \quad n \log \frac{u_n}{u_{n+1}} = n\left[1 - n\log\left(1+\frac{1}{n}\right)\right]$$

$$= n\left[1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \ldots\ldots\right)\right]$$

$$= n\left[1 - 1 + \frac{1}{2n} - \frac{1}{3n^2} + \frac{1}{4n^3} \ldots\ldots\right]$$

$$= \frac{1}{2} - \frac{1}{3n} + \frac{1}{4n^2} + \ldots\ldots\ldots\ldots$$

$$\lim_{n\to\infty}\left(n \log \frac{u_n}{u_{n+1}}\right) = \frac{1}{2} \text{ which is} < 1.$$

By log test, Σu_n is divergent for $x = \frac{1}{e}$.

Hence, the given series is convergent if $x < \frac{1}{e}$ and is divergent if $x \geq \frac{1}{e}$.

Q. 3. (a) Test for convergence the series $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n$.

Ans.Given :
$$u_n = \frac{(n!)^2}{(2n)!} x^n$$

$$u_{n+1} = \frac{\{(n+1)!\}^2 x^{n+1}}{(2n+2)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n!)^2 x^n}{(2n)!} \frac{(2n+2)!}{\{(n+1)!\}^2 x^{n+1}}$$

$$= \frac{1}{x} \cdot \frac{(2n+2)(2n+1)}{(n+1)^2}$$

$$\frac{u_n}{u_{n+1}} = \frac{4}{x} \frac{\left(1+\frac{1}{2n}\right)}{\left(1+\frac{1}{n}\right)}$$

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{4}{x}$$

Now, if $\frac{4}{x} > 1$, $x < 4$

$\lim \frac{u_n}{u_{n+1}} > 1$ and hence by ratio test the series converges.

Now, if $x > 4$

$\lim \frac{u_n}{u_{n+1}} < 1$, series diverges.

Now, if $x = 4$, ratio test fails

At $x = 4$

$$\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{1}{2n}\right)}{\left(1+\frac{1}{n}\right)}$$

$$= \frac{2n+1}{2(n+1)}$$

$$n\left(\frac{u_n}{u_{n+1}}-1\right)=n\left[\frac{2n+1}{2n+2}-1\right]$$

$$=n\left[\frac{-1}{2n+2}\right]$$

$$=-\frac{1}{2}\left[\frac{1}{1+\frac{1}{n}}\right]$$

$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=-\frac{1}{2}$ which is < 1, thus the given series is divergent at $x = u$.

Therefore, if $x \geq 4$, series diverges and if $x < 4$, series converges.

Q. 3. (b) Test the convergence of the series :

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sin n\alpha}{n^2},\ \alpha \text{ is real.}$$

Ans. Given series is,

$$(-1)^{n-1}\frac{\sin n\alpha}{n^2},\ \alpha \text{ is real.}$$

This is an alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}u_n,$$

Where $$u_n=(-1)^{n-1}\frac{\sin n\alpha}{n^2}$$

$$|u_n|=\left|(-1)^{n-1}\frac{\sin n\alpha}{n^2}\right|$$

$$=\frac{1}{n^2}|\sin n\alpha|$$

$$\leq\frac{1}{n^2}\quad\left[\because|\sin n\alpha|\leq 1\right]$$

Since, the series $\sum\frac{1}{n^2}$ is convergent, therefore by comparison test the series $\Sigma|u_n|$ is also convergent and hence the given series is absolutely convergent.

Unit-II

Q. 4. (a) For the matrix $A=\begin{bmatrix}1 & 1 & 2\\ 1 & 2 & 3\\ 0 & -1 & -1\end{bmatrix}$. **Find non-singular matrices *P* and Q S.t. PAQ is in the normal form.**

Ans. Given matrix
$$A=\begin{bmatrix}1 & 1 & 2\\ 1 & 2 & 3\\ 0 & -1 & -1\end{bmatrix}$$

$$A_{3\times 3}=I_3AI_3$$

$$\begin{bmatrix}1 & 1 & 2\\ 1 & 2 & 3\\ 0 & -1 & -1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}A\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$R_2\to R_2-R_1$$

$$\begin{bmatrix}1 & 1 & 2\\ 0 & 1 & 1\\ 0 & -1 & -1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}A\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$R_3\to R_3+R_2$$

$$\begin{bmatrix}1 & 1 & 2\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ -1 & 1 & 1\end{bmatrix}A\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$C_2\to C_2-C_1,\ C_3\to C_3-2C_1$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ -1 & 1 & 1\end{bmatrix}A\begin{bmatrix}1 & -1 & -2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$C_3\to C_3-C_2$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ -1 & 1 & 1\end{bmatrix}A\begin{bmatrix}1 & -1 & -1\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}$$

Hence, PAQ is in normal form, where

$$P=\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ -1 & 1 & 1\end{bmatrix},\quad Q=\begin{bmatrix}1 & -1 & -1\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}.$$

Q. 4. (b) For what values of k the equations :

$$x+y+z=1,\ 2x+y+4z=k,\ 4x+y+10z=k^2$$

have a solution and solve them completely in each case.

Ans. For the given system of equations, the augmented matrix $[A:B]$ is,

$$[A:B]=\begin{bmatrix}1 & 1 & 1 & : & 1\\ 2 & 1 & 4 & : & K\\ 4 & 1 & 10 & : & K^2\end{bmatrix}$$

$$R_2 \to R_2 - 2R_1$$
$$R_3 \to R_3 - 4R_1$$

$$\sim\begin{bmatrix}1 & 1 & 1 & : & 1\\ 0 & -1 & 2 & : & K-2\\ 0 & -3 & 6 & : & K^2-4\end{bmatrix}$$

$$R_3 \to R_3 - 3R_2$$

$$\sim\begin{bmatrix}1 & 1 & 1 & : & 1\\ 0 & -1 & 2 & : & K-2\\ 0 & 0 & 0 & : & K^2-3K+2\end{bmatrix}$$

(i) For the unique solution of the given system of equations,

$\rho(A)=\rho(A:B)=3$, which is not possible. Hence, there is no solution.

(ii) For infinite number of solution,

$$\rho(A)=\rho(A:B)<3$$

i.e., $K^2-3K+2=0$

$$K=2,1$$

Case 1 : When $K=2$

$$\begin{bmatrix}1 & 1 & 1\\ 0 & -1 & 2\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$

Let $$z = K_1$$

$$-y + 2z = 0$$

$$y = 2K_1$$

$$x + y + z = 1$$

$$x = 1 - 3K_1$$

Hence, $$x = 1 - 3K_1$$

$$y = 2K_1$$

$$z = K_1$$

Case 2 : When K = 1

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Let $$z = K_2$$

$$-y + 2z = -1$$

$$y = 1 + 2K_2$$

$$x = 1 - (y + z)$$

$$x = -3K_2$$

Hence, $x = -3K_2$, $y = 1 + 2K_2$, $z = K_2$ **Ans.**

Q. 5. (a) Find the eigen values and corresponding eigen vectors of the matrix :

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Ans. The characteristic equation of the given matrix is,

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)\left[9 + \lambda^2 - 6\lambda - 1\right] + 2(-6 + 2\lambda + 2) + 2(2 - 6 + 2\lambda) = 0$$

$$(\lambda - 2)^2 (\lambda - 8) = 0$$

$$\lambda = 2, 2, 8$$

The eigen values are 2, 2, 8.

Eigen vector for $\lambda = 2$ is,

$$[A - \lambda I][X] = [0]$$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $$2x_1 - x_2 + x_3 = 0$$

This equation is satisfied by

$$x_1 = 0,\ x_2 = 1,\ x_3 = 1$$

And $$x_1 = 1,\ x_2 = 3,\ x_3 = 1$$

Thus, eigen vectors are,

$$[0,\ 1,\ 1]^T,\ [1, 3, 1]^T$$

Eigen vector for $\lambda = 8$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $$x_1 + x_2 - x_3 = 0$$

And $$x_2 + x_3 = 0$$

Let $x_3 = 1,\ x_2 = -1,\ x_1 = 2$

Eigen vector is $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Thus, the three eigen vectors are,

$$[0, 1, 1]^T,\ [1, 3, 1]^T,\ [2, -1, 1]^T.$$

Q. 5. (b) Define similar matrices and discuss the nature of the quadratic form :

$$\mathbf{2x^2 + 2y^2 + 3z^2 + 2xy - 4yz - 4xz.}$$

Ans. Similar Matrices : A_1 square matrix A_2 of order n is called similar to a square matrix A of order n if,

$$A_1 = P^{-1}A_2P \text{ for some non-singular } n \times n \text{ matrix P.}$$

This transformation of a matrix A_2 by a non-singular matrix P to A_1 is called a similarity transformation.

The two similar matrices have the same eigen values.

The real symmetric matrix A associated with the given quadratic form $2x^2+2y^2+3z^2+2xy-4yz-4xz$ is

$$A=\begin{bmatrix}2 & 1 & -2\\ 1 & 2 & -2\\ -2 & -2 & 3\end{bmatrix}$$

Its characteristic equation is,

$$\begin{vmatrix}2-\lambda & 1 & -2\\ 1 & 2-\lambda & -2\\ -2 & -2 & 3-\lambda\end{vmatrix}=\lambda^3-7\lambda^2+7\lambda-1=0$$

$$\Rightarrow \qquad (\lambda-1)\left\{\lambda-\left(3+\sqrt{8}\right)\right\}\left\{\lambda-\left(3-\sqrt{8}\right)\right\}=0$$

The eigen values are

$$\lambda=1, 0.1715, 3.1715$$

Since, all eigen values are positive, so the given quadratic form is positive definite.

Unit-III

Q. 6. (a) If $y=e^{a\sin^{-1}x}$ prove that

$$\left(1-x^2\right)y_{n+2}-(2n+1)xy_{n+1}-\left(n^2+a^2\right)y_n=0, \text{ hence find } (y_n)_0.$$

Ans.

$$y=e^{a\sin^{-1}x} \qquad \text{...(1)}$$

$$y_1=e^{a\sin^{-1}x}\cdot\frac{a}{\sqrt{1-x^2}} \qquad \text{...(2)}$$

$$\left(1-x^2\right)y_1^2=a^2y^2$$

Again differentiating both sides,

$$\left(1-x^2\right)2y_1y_2-2xy_1^2=2a^2yy_1$$

$$\left(1-x^2\right)y_2-xy_1-a^2y=0 \qquad \text{...(3)}$$

Differentiating n times using Leibritz theorem,

$$\left(1-x^2\right)y_{n+2}+{}^nC_1(-2x)y_{n+1}+{}^nC_2(-2)y_n-xy_{n+1}-{}^nC_1y_n-a^2y_n=0$$

$$\left(1-x^2\right)y_{n+2}-[2xn-x]$$

$$y_{n+1}-n(n-1)y_n-^n y_n-a^2y_n=0$$

$$\boxed{\left(1-x^2\right)y_{n+2}-(2n+1)xy_{n+1}-\left(n^2+a^2\right)y_n=0} \qquad ...(4)$$

Putting x = 0 in equations (1), (2), (3) and (4),

$$y(0)=1,\ y_1(0)=a,\ y_2(0)=a^2$$

$$y_{n+2}(0)=\left(n^2+a^2\right)y_n(0) \qquad ...(5)$$

Putting n = 1, 2, 3, 4 ...... in equation (5)

$$y_3(0)=\left(1^2+a^2\right)y_1(0)$$

$$=a\left(1^1+a^2\right)$$

$$y_4(0)=\left(2^2+a^2\right)y_2(0)$$

$$=\left(2^2+a^2\right)a^2$$

$$y_5(0)=a\left(1^2+a^2\right)\left(3^2+a^2\right)$$

$$y_6(0)=a^2\left(2^2+a^2\right)\left(4^2+a^2\right) \text{ and so on}$$

In general,

$$y_n(0)=\begin{cases} a^2\left(2^2+a^2\right)\left(4^2+a^2\right)------\left\{(n-2)^2+a^2\right\}, n \text{ is even} \\ a\left(1^2+a^2\right)\left(3^2+a^2\right)--------\left\{(n-2)^2+a^2\right\}, n \text{ is odd} \end{cases}$$

Q. 6. (b) Show that radius of curvature ℓ at P on an ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is given by}$$

$\ell = CD^3/ab$ where CD is the semi-diameter conjugate to CP.

Ans. Let CP and CD be two conjugate diameters of the given ellipse.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

If the co-ordinates of P are given by t, i.e, (a cost, b sin t), then to find the radius of curvature 'ρ'.

$$\frac{dx}{dt} = -a\sin t\,,\ \frac{d^2x}{dt^2} = -a\cos t$$

$$\frac{dy}{dt} = a\cos t\,,\ \frac{d^2y}{dt^2} = -a\sin t$$

$$\rho = \frac{\left(1+\frac{d^2y/dx^2}{d^2x/dt^2}\right)^{3/2}}{\left(\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{d^2x}{dt^2}\right)}$$

$$\rho = \frac{\left(a^2\sin^2 t+b^2\cos^2 t\right)^{3/2}}{ab\sin^2 t+ab\cos^2 t}$$

$$\rho = \frac{\left(a^2\sin^2 t+b^2\cos^2 t\right)^{3/2}}{ab} \quad ...(1)$$

Also, we know that co-ordinates of D are given by $\left(\frac{\pi}{2}+t\right)$ i.e.,

$$\left\{a\cos\left(\frac{\pi}{2}+t\right), b\sin\left(\frac{\pi}{2}+t\right)\right\}$$

i.e., $(-a\sin t, b\cos t)$

Hence, $CD = \sqrt{a^2\sin^2 t+b^2\cos^2 t}$

From equation (1)

$$\rho = \frac{\left(\sqrt{a^2\sin^2 t+b^2\cos^2 t}\right)^3}{ab}$$

$$\rho = \frac{(CD)^3}{ab}$$ **Ans.**

Q. 7. (a) Given $x+y+z=a$, find the maximum value of $x^m y^n z^p$.

Ans. Let $u = x^m y^n z^p$ ...(1)

$$\phi(x,y,z) \equiv x+y+z = a \quad ...(2)$$

$$\log u = m\log x + n\log y + p\log z$$

$$\frac{1}{u}\frac{\partial u}{\partial x}=\frac{m}{x}$$

$$\frac{\partial u}{\partial x}=\frac{um}{x}$$

Similarly, $$\frac{\partial u}{\partial y}=\frac{nu}{y}$$

And $$\frac{\partial u}{\partial z}=\frac{pu}{z}$$

Given function is $\phi(x,y,z)=x+y+z-a$

Lagrange's equations are,

$$\frac{\partial u}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0$$

$$\frac{\partial u}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0$$

$$\frac{\partial u}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0$$

i.e., $$\frac{um}{x}+\lambda=0 \qquad ...(3)$$

$$\frac{nu}{y}+\lambda=0 \qquad ...(4)$$

$$\frac{pu}{z}+\lambda=0 \qquad ...(5)$$

From equations (3), (4) and (5)

$$x=-um/\lambda$$

$$y=-nu/\lambda$$

$$z=-pu/\lambda$$

Putting these values in equation (2)

$$x+y+z=a$$

$$-\frac{mu}{\lambda}-\frac{nu}{\lambda}-\frac{pu}{\lambda}=a$$

$$\frac{-u(m+n+p)}{a}=\lambda \Rightarrow -\frac{u}{\lambda}=\frac{a}{m+n+p}$$

Thus, $$x=\frac{am}{m+n+p}$$

$$y = \frac{an}{m+n+p}$$

$$z = \frac{ap}{m+n+p}$$ **Ans.**

Q. 7. (b) If $|a| < 1$, prove that : $\int_0^{\Pi} \log(1 + a\cos x)\,dx = \Pi \log\left[\frac{1}{2} + \frac{1}{2}\sqrt{1-a^2}\right]$.

Ans. $|a| < 1$, to prove

$$\int_0^{\pi} \log(1 + a\cos x)\,dx = \pi \log\left[\frac{1}{2} + \frac{1}{2}\sqrt{1-a^2}\right]$$

Let $$I = \int_0^{\pi} \log(1 + a\cos x)\,dx$$

Using IV property of definite integration

$$I = \int_0^{\pi} \log\{1 + a\cos(\pi - x)\}$$

Unit-IV

Q. 8. (a) Express $\int_0^1 x^m \left(1 - x^n\right)^p dx$ in terms of gamma functions and hence evaluate :

$$\int_0^1 x^5 \left(1 - x^3\right)^{10} dx$$

Ans. Given integral is,

$$I = \int_0^1 x^m \left(1 - x^n\right)^p dx$$

Let $$x^n = y,\ x = y^{1/n}$$

$$nx^{n-1}dx = dy$$

$$dx = \frac{dy}{nx^{n-1}} = \frac{dy}{ny^{(n-1)/n}}$$

Thus, $$I = \frac{1}{n}\int_0^1 y^{m/n} (1-y)^p\, y^{(1-n)/n} dy$$

$$= \frac{1}{n}\int_0^1 y^{\left(\frac{m+1}{n} - 1\right)} (1-y)^{p+1-1} dy$$

$$\boxed{I = \frac{\frac{1}{n}\Gamma\left(\frac{m+1}{n}\right)\Gamma(p+1)}{\Gamma\left(\frac{m+1}{n}+p+1\right)}}$$

Now for the given integral

$$\int_0^1 x^5\left(1-x^3\right)^{10} dx$$

$$m = 5, n = 3, p = 10$$

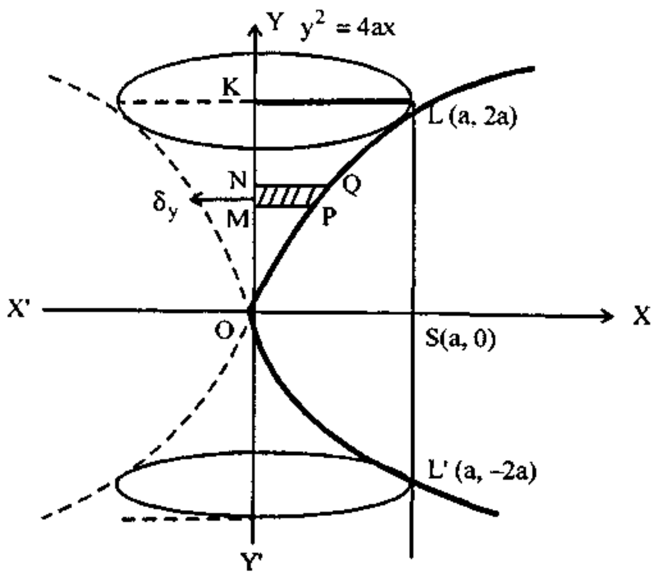
$$I = \frac{\frac{1}{3}\Gamma(2)\Gamma(11)}{\Gamma(13)} = \frac{\frac{1}{3}\times 1\times \Gamma(11)}{12\Gamma(11)}$$

$$I = \frac{1}{36}$$ **Ans.**

Q. 8. (b) Find the volume of reel-shaped solid formed by the revolution about the y-axis, of the part of parabola $y^2 = 4ax$ cut-off by the latus rectum.

Ans. The given parabola is :

$$y^2 = 4ax$$



The volume of the reel generated by the revolution of the arc cutt-off by the latus rectum LL' about y-axis.

= 2 × Volume generated by revolving the area OLK about y-axis.

Let us consider an elementary strip PMNQ parallel to the axis of x, where P is the point (x, y) and Q is the point $(x+\delta x, y+\delta y)$ on the parabola $y^2 = 4ax$.

Now, volume of the elementary disc formed by revolving the strip PMNQ about y-axis.

$$= \pi(PM)^2(NM) = \pi x^2 \delta y$$

Also length of the semi-latus rectum SL is 2a, therefore y varies from 0 to 2a.

$\therefore$ the required volume.

$$2\int_0^{2a} \pi x^2 . \delta y$$

$$= 2\pi\int_0^{2a} \frac{y^4}{16a^2} \delta y \quad \left[\because y^2 = 4ax\right]$$

$$= \frac{\pi}{8a^2}\left[\frac{y^5}{5}\right]_0^{2a} = \frac{\pi}{40a^2}\left(32a^5\right)$$

Required volume $= \dfrac{4\pi a^3}{5}$ **Ans.**

Q. 9. (a) Find by double integration, the area laying inside the circle $r - a\sin\theta$ and outside the cardioid $r = a(1-\cos\theta)$.

Ans. Given curves are

$$r = a\sin\theta \quad ...(1)$$

$$r = a(1-\cos\theta) \quad ...(2)$$

On solving equations (1) and (2),

$$\sin\theta = 1 - \cos\theta$$

$$\sin\theta + \cos\theta = 1$$

Squaring both sides,

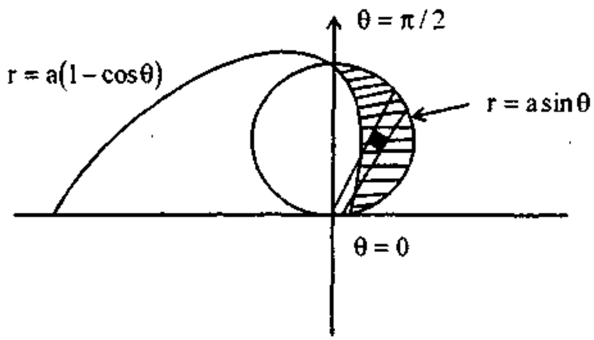
$$\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = 1$$

$$\sin 2\theta = 0$$

$$2\theta = 0$$

$$\theta = 0 \text{ or } \pi/2$$

Required area is the shaded portion.



r varies from $a(1-\cos\theta)$ to $a\sin\theta$ and θ varies from 0 to $\pi/2$.

Required area $$= \int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi/2}\left[r^2\right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\left[\sin^2\theta-(1-\cos\theta)^2\right]d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}(-1-\cos 2\theta+2\cos\theta)d\theta$$

$$= \frac{a^2}{2}\left[-\theta-\frac{\sin 2\theta}{2}+2\sin\theta\right]_0^{\pi/2} = \frac{a^2}{2}\left[-\frac{\pi}{2}+2\right]$$

Required area $$= a^2\left(1-\frac{\pi}{4}\right)$$ **Ans.**

Q. 9. (b) Using triple integration, find the volume of the sphere $x^2+y^2+z^2=a^2$.

Ans. Given sphere is $x^2+y^2+z^2=a^2$

Required volume $$= \iiint_S dz\,dy\,dx$$

$$V = 8\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$$

$$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx$$

$$= 8\int_0^a\left[\frac{y}{z}\sqrt{a^2-x^2-y^2}+\frac{(a^2-x^2)}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$V = 8\int_0^a + \frac{(a^2-x^2)}{2}\frac{\pi}{2}dx$$

$$= +\frac{8\pi}{4}\left[a^2x-\frac{x^3}{3}\right]_0^a = +2\pi\left[a^3-\frac{a^3}{3}\right] = 2\pi\left(\frac{2a^3}{3}\right)$$

$$\boxed{V = \frac{4}{3}\pi a^3}$$ **Ans.**