5) Dice sum & Digit sum :-

:- sum of 3 dice equal to 12

$$x_1 + x_2 + x_3 = 12$$

$1 \le x_1 \le 6 \Rightarrow 1 \le x_2 \le 6,\ 1 \le x_3 \le 6$

:- If the number is less than '6' so we dont require upper constraint as dice can go max to '6' so lower constraint can we adjusted with $^{n-1+r}C_r$ upto '6' upper constraint, dice problems like:

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$x_1 + x_2 + x_3 \le 6$

$x_1 + x_2 + x_3 = 5$ . . etc.

Digit sum :-

upto '9' can be solved with $^{n-1+r}C_r$

like:

Que: ? How many number have digit sum = 9

$$x_1 + x_2 + x_3 = 9$$

for three digit numbers lower constraint = 1 atleast

so $\boxed{^{3-1+8}C_8}$ 9-(1) lower constraint

:- for 3 digit string, we don't need to set lower ~~b~~ constraint to 1, it can be '0' so

$$x_1 + x_2 + x_3 = 9$$

$\boxed{3-1+9_{C_9}}$ ← for strings

eg:-

1. $x_1 + x_2 + x_3 = 10$ $\quad 3-1+10_{C_{10}} = \underline{66}$

$0 \leq x_1, x_2, x_3$

2. $x_1 + x_2 + x_3 = 1$

Que. n identical balls into k boxes so each box contain atleast 1 ball

$x_1 + x_2 - - - + x_n =$

3.) $x_1 + x_2 + x_3 = 10$

$-3 \le x_1 , \quad -2 \le x_2 , \quad 1 \le x_3$

$x_1 + x_2 + x_3 = 10 - (-4) = 14$

$3 - 1 + 14_{C_{14}}$ =

4.) Upper constraint Problem :-

:- If only one upper Constraint then It can be adjusted.

$x_1 + x_2 + x_3 = 10$

take approix $x \ge 8$ $\leftarrow$ $x_1 \le 7 \quad x_2 \ge 0 \quad x_3 \ge 0$

$(3-1+10_{C_{10}}) - (3-1+2_{C_2})$

:- multiple upper Constraint
upper Constraint & lower Constraint } Generating functn.

5.) $x_1 + x_2 + x_3 \le 10$

ways to put 10 balls to put atmost into 3 boxes.

for atmost Problem, add 1 more box.

$x_1 + x_2 + x_3 + x_4 = 10$ so $4 - 1 + 10_{C_{10}}$

3.) $x_1 + x_2 + x_3 = 10$

$-3 \leq x_1 , \quad -2 \leq x_2 , \quad 1 \leq x_3$

$x_1 + x_2 + x_3 = 10 - (-4) = 14$

$3 - 1 + {}^{14}C_{14} =$

4) <u>upper constraint Problem</u> :-

:- If only one upper constraint then it can be adjusted.

$x_1 + x_2 + x_3 = 10$

take opposite $x \geq 8$ ← $x_1 \leq 7$ $\quad x_2 \geq 0 \quad x_3 \geq 0$

$({3-1+10}C_{10}) - (3-1+{}^{2}C_2)$

:- multiple upper constraint
upper constraint & lower constraint } Generating funct^n.

*5.) $x_1 + x_2 + x_3 \leq 10$

ways to put 10 balls to put atmost into 3 boxes.

for atmost Problem, add 1 more box.

$x_1 + x_2 + x_3 + x_4 = 10$ so $4-1+{}^{10}C_{10}$

6.) $x_1 + x_2 + x_3 = 10$ id Red Balls

$= 15$ id green Balls

$= 12$ id Blue Balls

Divide it into tasks

$t_1 \quad t_2 \quad t_3$

$$\boxed{3-1+{}^{10}C_{10} \times 3-1+{}^{15}C_{15} \times 3-1+{}^{12}C_{12}}$$

Que:-

2 girls are picking:

$\underbrace{\text{10 Lillies}}_{\text{white}}$, $\underbrace{\text{12 sunflowers}}_{\text{yellow}}$, $\underbrace{\text{15 daffodils}}_{\text{one colour}}$

:- so they are identical

In How many ways girls can distribute the flowers b/w themselves.

$\cancel{2-1 \times 1}$ $\underset{t_1}{(2-1+{}^{10}C_{10})} \times \underset{t_2}{(2-1+{}^{12}C_{12})} \times \underset{t_3}{(2-1+{}^{15}C_{15})}$

Combination with limited Repetition :-

multiset :

:— $\{a*3, b*4, c*5\}$

$$\underline{x_a + x_b + x_c = 10}$$

Proof :- $\underline{{}^{n-1+r}C_r}$

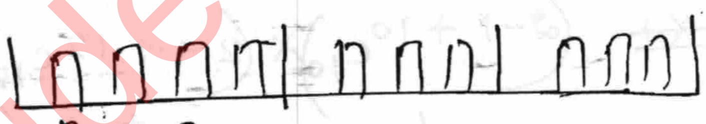
let $x_1 + x_2 + x_3 = 10$

0 0 0 | 0 0 | 0 0 0 0 0

2 1's → so $(n-1)$ 1's r 0's

$$\boxed{{}^{n-1+r}C_r}$$

:— 10 books with 3 shelves.



$\underbrace{B_1\ B_2\ B_3}_{\text{identical Books}}$

so ${}^{n-1+r}P_r$ ← shelves

so $\left({}^{3-1+10}P_{10}\right)$

:— 10 rings in left hand with 5 fingers

${}^{n-1+r}P_r$ so ${}^{5-1+10}P_{10}$

7. Distribution Problem :-

distributing objects (distinct objects)

distribute — Divide

D — D

(ordered Partition) $\downarrow$ op — $\downarrow$ unordered Partition (up)

Distinct objects — IDentical objects

:- D $\xrightarrow{\text{op}}$ D (ordered Partition)

1.) Distribution fully specified :-

10 toys → A: 3, B: 3, C: 4

$$\frac{10!}{3!\,3!\,4!}$$

2.) Distribution unspecified :-

10 toys → 3 children

$$3^{10}$$

3.) Distribution partialy specified :-

5 toys → A: ≥ 1, B: ≥ 1, C: ≥ 1

$$\Rightarrow \boxed{\frac{5!}{3!}\times 3+\frac{5!}{2!\,2!}\times 3}$$

first allot 1, 1, 1

5 → A B C

A	B	C
3	1	1
1	3	1
1	1	3
2	2	1
2	1	2

eg:-

20 toys

A B C

≥ 1

} universal set $= \underline{3^{20}}$

take compliment

$= 0$ so it will

$\underline{2^{20}}$

$\Rightarrow \boxed{3^{20} - 2^{20}}$

20 toys

A B C

≥ 2

take complimt 1, or 0

0 only so $\left(2^{20} + \underbrace{20C_1 * 2^{19}}_{\text{exact 1 toys for A}}\right)$

so $\boxed{3^{20} - \left[2^{20} + 20C_1 * 2^{19}\right]}$

unordered partition :-

fully specified Only :-

9 toys

3 3 3

A B C

$$\frac{9!}{3!\,3!\,3!\,3!}$$

$3! \rightarrow$ because its unordered

eg:

10

3 3 2 2

$\underbrace{3 \quad 3}_{\text{only can be exchanged}}$ $\underbrace{2 \quad 2}_{\text{only can be exchanged}}$

$$\Rightarrow \frac{10!}{(3!)^2 (2!)^2 \, 2! \, 2!}$$

eg:

~~10~~ 7

4 1 1 1

$\underbrace{1 \quad 1 \quad 1}$

$$\Rightarrow \frac{7!}{4! \, 3!}$$

eg:

40 ~~tea~~ People

↓

22 (2 teams)

as No name to team

11 11

$\underbrace{11 \quad 11}_{\text{can be exchanged}}$

~~48~~

$$\Rightarrow {}^{40}C_{22} \times \frac{22!}{(11!)^2 \, 2!}$$

Inclusion & Exclusion :-

2 set

$$\left\{\begin{array}{l}
n(A \cup B) = n(A) + n(B) - n(A \cap B) \quad \{\text{either by } A \text{ or } B\} \\
n(A \oplus B) = n(A) + n(B) - 2\cancel{\times}[n(A \cap B)] \\
n(A - B) = n(A) - n(A \cap B) \quad \{\text{divisible by } A \text{ but not } B\} \\
n(B - A) = n(B) - n(A \cap B) \\
n(A^c \cap B^c) = n(U) - [n(A) + n(B) - n(A \cap B)] \\
\qquad \text{neither by } A \text{ \& nor by } B \\
n(A^c \cup B^c) = n(U) - n[A \cap B] \\
\qquad \text{either not by } A \text{ or not by } B
\end{array}\right.$$

Required
$$\left\{\begin{array}{ll}
n(A) & = 2^6 \\
n(B) & = 2^5 \\
n(A \cap B) & = 2^3 \\
n(U) & = 2^8
\end{array}\right\} \Downarrow$$

_ _ _ _ _ _ _ _

1 1 _ _ _ _ _ _

_ _ _ _ _ 0 0 0

1 1 _ _ _ 0 0 0

eg:- 1 to 100 Numbers

$\div 2$ or $\div 3$ $\qquad n(U) = 100$

$A : \div 2 \Rightarrow n(A) = \left\lfloor \frac{100}{2} \right\rfloor = 50$

$B : \div 3 \Rightarrow n(B) = \left\lfloor \frac{100}{3} \right\rfloor = 33$

floor

$n(A \cap B) = \left\lfloor \frac{100}{6} \right\rfloor = 16$

3-set :-

$\rightarrow$ $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

$\star$ $n(A \oplus B \oplus C) = n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(A \cap C) - 2n(B \cap C) + 3n(A \cap B \cap C)$

$\rightarrow$ $n(\bar{A} \cap \bar{B} \cap \bar{C}) = n(U) - n(A \cup B \cup C)$

$\rightarrow$ $n(\bar{A} \cup \bar{B} \cup \bar{C}) = n(U) - n(A \cap B \cap C)$

Que:- $1 - 100$ $\qquad n(A) = \frac{100}{2}, \; n(B) = \frac{100}{3}, \; n(C) = \frac{100}{5}$

2, 3, 5

$n(\bar{A} \cap \bar{B} \cap \bar{C}) = n(U) - n(A \cup B \cup C)$

$= 100 - [$

"Drangement" :-

Que:-

1	2	3	4	5

(1) (2) (3) (4) (5)

Ball i is not in Box i $\quad i = 1, 2, 3, 4, 5$

:- Not any one number should not be in its sequence. (Position)

$A = \{1, 2, 3, 4, 5\}$ $\qquad$ 3, 5, 4, 1, 2 for this :-

$$n(\bar{A}\cap\bar{B}\cap\bar{C}\cap\bar{D}\cap\bar{E}) = n(U) - n(A\cup B\cup C\cup D\cup E)$$

$$= 5! - \big(n(A)+n(B)+n(C) + n(D)+n(E) + \ldots - n(A\cap B) - n \ldots\big)$$

$$= 5! - [5C_1 \times 4! - 5C_2 \times 3! + 5C_3 \times 2! - 5C_4 \times 1! + 5C_5 \times 0!]$$

$$= 120 - 76 = \underline{44}$$

formula Pattern :

$$= n! - [nC_1 \times (n-1)! - nC_2 \times (n-2)! + \ldots - nC_n \times 0!]$$

$$= \cancel{n!} - \left[\cancel{n!} - \frac{n(n-1)}{2!} \times (n-2)! + \ldots\right]$$

$$= \left[+\frac{n!}{2!} - \frac{n!}{3!} + \frac{n!}{4!} - \frac{n!}{n!}\right]$$

$$= \sum_{r=2}^{n} (-1)^r \frac{n!}{r!}$$

$$\boxed{D_n \Rightarrow \sum_{r=2}^{n} (-1)^r \frac{n!}{r!}}$$

so $D_5 = \sum_{2}^{5} (-1)^r \frac{5!}{r!}$

$= \frac{5!}{2!} - \frac{5!}{3!} + \frac{5!}{4!} - \frac{5!}{5!}$

$= 60 - 20 + 5 - 1 = 44$

2. all Balls on its right position $= \underline{1}$

★ 3. atleast one ball in right position

$\Rightarrow \underline{n! - D_n}$

4. atleast one ball on its right position

$\Rightarrow \underline{n! - 1}$

eg:- 5 CPU

□ □ □ □ □

1 – connect every CPU to wrong socket

$= \underline{D_5}$ (Derangement)

Pigeon-Hole Principle :- "same" ⇒ "Pigeon Hole"

If $\boxed{n}$ Pigeons occupy m pigeon-holes and $n > m$

then

$\geq \left(\left\lfloor \frac{n-1}{m} \right\rfloor + 1 \right)$ Pigeon sharing the same hole

eg:- If 50 are painted with 7 colors

atleast ? will have same color ?

$\boxed{1 \geq ?}$ ← key word

$n = 50$
$m = 7$

$\geq \left(\left\lfloor \frac{50-1}{7} \right\rfloor \right) + 1 = \underline{8}$

eg:- If 10 cards delt from 52 Cards deck

$\geq$? "same" suit ?
P.H.

$n = 10$
$m = 4$ suits are 4 in cards

$\left\lfloor \frac{10-1}{4} \right\rfloor + 1 = \left\lfloor \frac{9}{4} \right\rfloor + 1 = 2+1 = 3$

↑ floor value

eg:- ~~atleast~~ minimum How many cards are the delt so atleast 3 of them from same suit

⇓

4 suits

n = ?

$$\Rightarrow \left\lfloor \frac{n-1}{4} \right\rfloor + 1 = 3$$

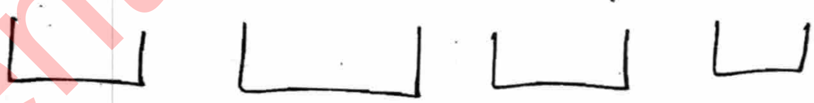
so $n = 9$ or 10 (approx value)

minimum

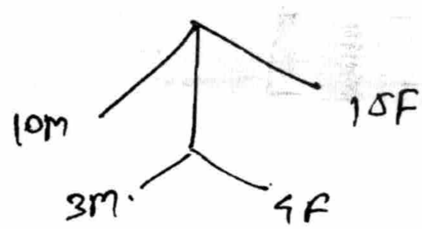
★ Note:-

$$\boxed{\left\lceil \frac{n}{m} \right\rceil = \left\lfloor \frac{n-1}{m} \right\rfloor + 1}$$

Que:- If every possible committee of 3M, 4F made by 10M, 15F is written on chits



Hats



$$n = {}^{10}C_3 \times {}^{15}C_4$$

$$m = 4$$

$$\Rightarrow \left(\left\lfloor \frac{{}^{10}C_3 \times {}^{15}C_4}{4} \right\rfloor + 1 \right)$$

Que: $A = \{1, 2, 3, 4, 5\}$

$\{1, 2\} = 3$

$\{4, 5\} = 9$

$m = 9 - 3 + 1 = 7$

$n = {}^5C_2 =$

$\geq \left\lfloor \frac{{}^5C_2}{7} \right\rfloor + 1$

Que- 10 Pairs - green

20 Pairs - Blue

15 Pair - Red

gaurantee to get green pair

$\Rightarrow \underline{40 + 30 + 10} = 80 + 1 = 81$ picking attempts

Summation :-

1. Bionomial summation $\rightarrow \sum_{r=2}^{n} r.{}^nC_r$
2. Generating function $\rightarrow$

1.) Binomial Summation :-

Property of Binomial summation :-

Binomial cofficient $= (n+1) \Rightarrow {}^nC_0, {}^nC_1, {}^nC_2 \ldots {}^nC_n$

$$(x+y)^n = \sum_{r=0}^{n} \underbrace{\boxed{{}^nC_r}}_{B.C.} x^r y^{n-r}$$

:- $(x+y)^{10} \Rightarrow \underbrace{B.C.?}_{{}^{10}C_2} x^2 y^8$

:- $(x+y)^{10} \Rightarrow \underbrace{B.C.}_{0} x^2 y^9$

:- $(x+y+z)^{10} \Rightarrow \underbrace{BC}_{\frac{10!}{2!\,3!\,5!}} \underline{x^2 y^3 z^5}$ $2+3+5=10$

$(x+y+z+T)^{10} \rightarrow \underline{}\ x^2 y^3 z^5 T^0$

B.C $\Rightarrow \dfrac{10!}{2!\,3!\,5!}$

$\Rightarrow \quad (x+y+z)^n = \sum \frac{n!}{n_1!\,n_2!\,n_3!} x^{n_1} y^{n_2} z^{n_3}$

$n_1 + n_2 + n_3 = n$

$\Rightarrow$ How many B.C..

$(x+y+z)^n = ?$

$\Rightarrow \quad 3-1+{}^{10}C_{10} = \underline{66}$

$\Rightarrow \quad (x_1 + x_2 + x_3 - - x_r)^n$

$\Rightarrow \quad \boxed{\cancel{n-1+{}^rC_r}}$

$\Rightarrow \quad \boxed{r-1+{}^nC_n}$

If $\left(x^2 + \frac{1}{x^3}\right)^{10} \Rightarrow$ B.C. ? x^{10}

$\Rightarrow \sum_{r=0}^{10} {}^{10}C_r (x^2)^r \left(\frac{1}{x^3}\right)^{10-r}$

$\Rightarrow \sum_{r=0}^{10} {}^{10}C_r \frac{x^{2r}}{x^{30-3r}}$

$\Rightarrow \sum_{r=0}^{10} {}^{10}C_r \, x^{5r-30}$

$\Rightarrow 5r - 30 = 10$

$\Rightarrow 5r = 40$

$r = 8$

If r = fraction number $= 8.2$ then $r = 0$

so $({}^{10}C_8) = 45$

Properties :-

1. Symmetric Property :-

:- $\boxed{{}^nC_r = {}^nC_{n-r}}$

2. Pascal's identity :-

:- $\boxed{{}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r}$

3. Newton's identity :-

:- $\boxed{{}^nC_r \cdot \times {}^rC_k = {}^nC_k \times {}^{n-k}C_{r-k}}$

4. Row - summation :-

:- $\boxed{\sum_{r=0}^{n} {}^nC_r = 2^n}$

5. Row - square summation :-

:- $\boxed{\sum_{r=0}^{n} ({}^nC_r)^2 = {}^{2n}C_n}$

6. Alternative sign Row summation :-

$\boxed{\sum_{r=0}^{n} (-1)^r \, {}^nC_r = 0}$

7. Even sum & odd summation :-

${}^nC_0 + {}^nC_2 + {}^nC_4 - -$

${}^nC_1 + {}^nC_3 + {}^nC_5 -$

$\boxed{\sum_{n=0}^{n} {}^nC_{2k} = \sum {}^nC_{2k-1} = 2^{n-1}}$

8. <u>Coloum Summation</u> :-

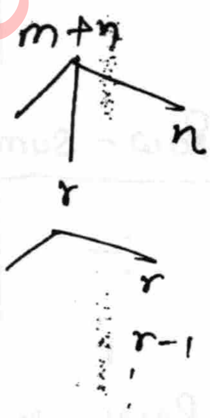
$$\boxed{\sum_{k=0}^{N} {}^{k}C_r = {}^{N+1}C_{r+1}}$$

9.

$$\boxed{\sum_{r=0}^{n} r \cdot {}^{n}C_r = n \cdot 2^{n-1}}$$

10. <u>VANDER MONDE'S IDENTITY</u> :-

$$\boxed{\sum_{k=0}^{r} {}^{m}C_k \, {}^{n}C_{r-k} = {}^{m+n}C_r}$$



:- <u>Pascal</u> :-

add one

$${}^{100}C_{30} + {}^{100}C_{71} = {}^{101}C_{71} = {}^{101}C_{30}$$

Bijone

$$\boxed{{}^{n}C_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r}$$

x included x excluded

$$\boxed{{}^{n}C_r = {}^{n-2}C_{r-2} + 2\,{}^{n-2}C_{r-1} + {}^{n-2}C_r}$$

x, y both included x (or) y included Both x, y excluded.

Newton's :-

(n) People
↓
(r) select
↓
k ⟶ leaders

$^{100}C_{90} \times {}^{90}C_{20}$

$= {}^{100}C_{20} \times {}^{80}C_{70}$

:- $^{64}C_{16} \times {}^{16}C_8 \Rightarrow {}^{64}C_8 \times {}^{56}C_8$

Row Sym:

$$\boxed{\sum_{r=0}^{n} {}^{n}C_r}\, x^r y^{n-r} = (x+y)^n$$

$(1+1) \Rightarrow 2^n$

$$\sum_{r=0}^{10} (-3)^r\, 6^{10-r}\, {}^{10}C_r = ?$$

$= (-3+6)^{10}$

$(x+y)^n$ Binomial sym.

eg:- $A = \{1, 2, 3, \ldots 10\}$

How many subsets of set includes 3 & 7

_ _ _ 7 3 _ _ _ _ _ _

$\Rightarrow 2^8$

:- Row Square :-

$$\sum_{r=0}^{10} {}^{10}C_r^2 = {}^{20}C_{10}$$

$$\sum_{r=1}^{10} {}^{10}C_r = 2^{10} - 1$$

Vander Mode's

$$\sum_{i=1}^{8} {}^{10}C_i \; {}^{15}C_{8-i} = {}^{25}C_8$$

Generating function :-

$$\boxed{A(x) = \sum_{r=0}^{\infty} a_r x^r}$$

dummy

Generating function

Sequence $a_0, a_1, a_2 --$	Numeric function $[a_r]_0^\infty$	Generating function $A(x)$
	differential / summation / integral	
1, 1, 1, 1, 1, 1 --	1^r	$\frac{1}{1-x}$
1, -1, 1, -1, - -	$(-1)^r$	$\frac{1}{1+x}$
1, a, a^2 - - - -	a^r	$\frac{1}{1-ax}$
1, 1/2, 1/4, 1/8 - -	$\left(\frac{1}{2}\right)^r$ or $\left(\frac{1}{a}\right)^r$	$\frac{2}{2-x} \Rightarrow \frac{a}{a-x}$ or $\frac{1}{1-\frac{1}{a}x}$
1, x, a^2, $-a^3$ --	$(-1)^r a^r$	$\frac{1}{1+ax}$
$1 \cdot x^0 + x^2 + x^4 + --$	$\sum_{r=0}^{\infty} x^{2r}$	$\frac{1}{1-x^2}$
1, 0, 1, 0, 1, 0 --	$a_r = 1, \; r = 2k$ (even); $= 0, \; r \neq 2k$ (odd)	$\frac{1}{1-x^2}$
1, 000, 1, 000, 1, -	$a_r = 1, \; r = 4k$	$\frac{1}{1-x^4}$

Note :-

1. $a_r \longrightarrow A(x)$

$k \cdot a_r \longrightarrow k\, A(x)$

2. $a_r \longrightarrow A(x)$

$b_r \longrightarrow B(x)$

$$\sum_{r=0}^{\infty} x^r = \frac{1}{1-x}$$

3. $a_r \longrightarrow A(x)$

$(-1)^r a_r \longrightarrow A(-x)$

$$\sum_{r=0}^{\infty} x^{2r} = \frac{1}{1-x^2}$$

~~$a_r \longrightarrow A(-x)$~~

Qu

$$\sum_{r=0}^{\infty} \left(3 \left(\frac{1}{2}\right)^r + 5 \cdot 7^r \right) x^r$$

$\left(\frac{1}{a}\right)^r$ $\quad a^r$

$$3 \times \frac{1}{1-\frac{1}{2}x} + 5 \cdot \frac{1}{1-7x}$$

$$\frac{6}{2-x} + \frac{5}{1-7x} \Rightarrow \frac{6(1-7x)+5(2-x)}{(2-x)(1-7x)}$$

eg :-

★ $\sum_{r=0}^{\infty} {}^{r+2}C_r \, x^r =$

$\Downarrow$

$$\sum_{r=0}^{\infty} {}^{3-1+r}C_r \, x^r = \frac{1}{(1-x)^3}$$

Sequence	Numeric function	generating function
* $^{n-1}C_0, {}^{n}C_1, {}^{n+1}C_2 \dots$	$^{n-1+r}C_r$	$\frac{1}{(1-x)^n}$
* $^{n}C_0, {}^{n}C_1, {}^{n}C_2 \dots$	$^{n}C_r$	$(1+x)^n$

* Shifting property of generating functn :-

$1, 1, 1, 1, 1, \dots \Rightarrow \frac{1}{1-x}$

$0, 1, 1, 1, 1 \dots \Rightarrow \frac{x}{1-x}$

$0, 0, 1, 1, 1 \dots \Rightarrow \frac{x^2}{1-x}$

$\Downarrow$

$a_r = 1 \quad r \geq 2$

$= 0 \quad r = 0, 1$

** $\frac{x^2}{(1-x)^{10}} \Rightarrow a_r = {}^{n-1+r}C_r \quad r \geq 2$

$= 0 \quad r = 0, 1$

:- $0, 0, {}^{9}C_0, {}^{10}C_1, {}^{10}C_2 \dots$