Set-Theory :-

1. Relation
2. Functions
3. Group theory
4. Po-sets, Lattices & BA

★ $\forall x\,(P(x) \to \varphi) \Rightarrow (\forall_x P(x)) \to \varphi$

Not true $\times \Leftarrow$

$\exists x\,(\alpha(x) \to \beta) \Leftarrow (\exists_x \alpha(x)) \to \beta$

$\Rightarrow \times$ Not true

$\forall x, \alpha(x) \to \beta = \forall_x(\alpha(x) \to \beta)$

comma   Bracket

** v. Imp

1. Relations :-

1. Defination
2. Domain & Range
3. R-relative set
4. Representation of Relations
5. operations on Relations :- $\{\cup, \cap, \bar{R}, R-S, S-R, R \oplus S\}$, $\{R^{-1}, S^{-1}, R\cdot S, S\cdot R\}$ R^2, S^2, R^n, S^n
6. * Types of Relations
7. * counting of Relations
8. * closures.
9. Equivalence & Partial order Relation

   (R S T)   (R AS T)
10. Properties of equivalence Relation :
11. R^n, R^*, R^+ :

(Binary Reln)

Defn:- Relation from A to B is $\subseteq A \times B$

cartesian Product

$\underline{A \times B} \rightarrow$ universal Relation (Biggest Reln)

$\phi \rightarrow$ Smallest Reln (Null Reln)

$A \times B = \{(x, y) \mid x \in A, \& \ y \in B\}$

$A = \{1, 2, 3\}$

$B = \{a, b\}$

It's an ordered Pair :- $A \times B = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$

$B \times A = \{(a,1), (b,1), (a,2), (b,2), (a,3), (b,3)\}$

$\Rightarrow \boxed{|A \times B| = |B \times A| = |A| \times |B|}$ {True}

$|A| = m$

$|B| = n$ size of largest Reln $= mn$

smallest Reln $= 0$ becoz $|\phi| = 0$

# of Reln $= 2^{mn}$

$1 R_a \Rightarrow (1, a) \in R$

$1 R_c \Rightarrow (1, c) \notin R$

:- $xRy \Rightarrow (x,y) \in R$

:- **Domain & Range :-**

Domain(R) = set of 1st element

Range(R) = set of 2nd element

$A = \{1, 2, 3\}$

$B = \{a, b\}$

$R = \{(1,a), (2,b), (2,a)\}$

Domain (R) = $\{1, 2\}$

Range (R) = $\{a, b\}$

If $R = \{A \times B\}$

$$\left\{ \begin{array}{l} \text{Domain}(R) \subseteq A \\ \text{Range}(R) \subseteq B \end{array} \right\}$$

eg :- $T = \{(x,y) \mid y = 3x + 1\}$ on $A \times B$ $R \times R$

$x = \frac{y-1}{3}$.

Dom (T) = Real No.

$\boxed{\text{Range}(T) = \text{Domain}(T^{-1})}$ = Real No.

eg:- $T = \{(x,y) \mid y = 3x+1\}$ on $Z \times Z \rightarrow$ int

$x = \frac{y-1}{3}$

$Dom(T) = Z$

$Range(T) = \{x \in Z / x \bmod 3 = 1\}$

$= (\ldots -8, -5, -2, -1, 4, 7, 10 \ldots)$

eg:-

$T = \{(x,y) / y = x^2\}$ on $R \times$ Real

$x = \sqrt{y}$

$Dom(T) = R$

$Range = R^+ \cup \{0\}$

eg:

$T = \{(x,y) / y = \log x]$ on $R \times$ Real

$\log 0 = -\infty$

$x = e^y$ → always real No.

$Dom(T) = R^+$

$Range(T) = R$

eg:

$\{(x,y) / y = 1/\cancel{x}\ 2-x\}$ on $R \times R$ ↙ default

$2-x = 1/y$

$Dom(T) = \{R - \{2\}$

$Range(T) = R - \{0\}$

:- ~~R-Relation~~

:- R- Relative sets :-

$A = \{1, 2, 3\}$ $B = \{a, b\}$

$R = \{(1,a), (2,a), (2,b), (3,a)\}$

$\forall x \in A$

$R(x) = \{y \mid xRy\}$

$R(1) = \{a\}$

$R(2) = \{a, b\}$

$R(3) = \{a\}$

$C = \{1, 2\}$

$C \subseteq A$

$R(C) = R(1) \cup R(2)$

$= \{a, b\}$

$$\boxed{R(C) = \bigcup_{\forall x \in C} R(x)}$$

:- $\boxed{R(A) = \text{Range } R}$

:- $\boxed{R^{-1}(B) = \text{Dom}(R)}$

:- $\boxed{R(\text{Dom}(R)) = \text{Range}(R)}$

:- $R^{-1}(\text{Range}(R)) = \text{Dom}(R)$

:- $R^{-1}(B) \subseteq \text{Dom}(R)$

:- $R^{-1}(R(x)) \neq x$

Representation of Relation :-

1. Listing
2. Statement ⇒ $x R y$ iff $x \leq y$
3. Set Builder
4. Matrix
5. Digraph
6. Arrow diagram
7. Table
8. graph

$\{(x,y) \mid x \leq y\}$ on A

$A = \{1, 2, 3\}$

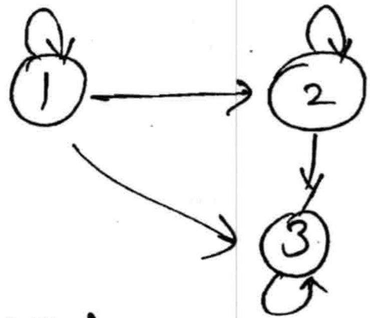
List :- $\{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$

matrix :-

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \Leftarrow \text{Boolean matrix}$$

Digraph :-

It is suitable for only $A \times A$



Arrow - diagram :-

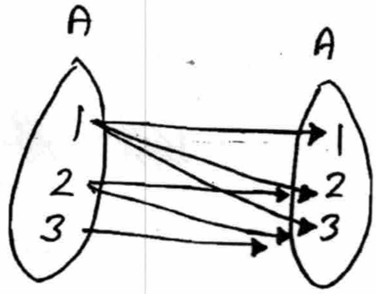


Table :- only for finite

x	y
1	1
1	2

graph :- suitable for real Numbers

$\{(x, y) \mid y = x^2\}$ in R

:- ~~Types of Relation~~ :-

:- "operation on R" :-

$R \cup S$, $R \cap S$, $\bar{R}$, $\bar{S}$, $R - S$, $S - R$, $R \oplus S$

$A = \{1, 2, 3\}$

$R = \{(1,1), (2,3), (3,1)\}$

$S = \{(3,1), (3,2), (1,3)\}$

$R \cup S = \{(1,1), (2,3), (3,1), (3,2), (1,3)\}$

$R \cap S = \{(3,1)\}$

$\bar{R} = A \times A - R$

$\bar{R} = \{(1,2)\ (1,3), (2,1), (2,2), (3,2), (3,3)\}$

eg:. Note:-

★ $|A| = n$

$|R| = x$

$|\bar{R}| = n^2 - x$

or

$|A|^2 - R$

eg:- $R - S = R - (R \cap S) = R \cap \bar{S}$

$S - R = S - (S \cap R)$

$R \oplus S = (R - S) \vee (S - R)$

$= (R \vee S) - (R \cap S)$

Note: $|R^{-1}| = |R|$

$R^{-1} \neq R$

Note:- composition

$R \cdot S(x) = R(S(x))$

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Types of Relations :-

1.) Reflexive :-

$R \ldots A \times A$

$A = \{1, 2, 3\}$

$R :$ Iff $\forall x \in A \quad xRx$

$R\{(1,1), (2,2), (3,3), (1,3)\}$ on A

2) IR-Reflexive :-

$R \{(1,3)(3,2), (2,3)\}$ on A

$IR :$ Iff $\forall x \in R \quad x \not R x$

eg: Note :-

1.) $R \Rightarrow$ not IR

2.) $IR \Rightarrow$ not R

3.) not $R \Rightarrow IR$ ✗

4) not $IR \Rightarrow R$ ✗

3) Symmetric :-

$\forall x, y \in A \quad xRy \Leftrightarrow yRx$ $\qquad R = \{(1,3), (3,2), (2,3), (3,1)\}$

Anti-Symmetric :- 1) $\boxed{xRy \ \& \ yRx \Rightarrow x = y}$

2) $\forall x, y \in A$ $\boxed{xRy \Rightarrow y\not{R}x \text{ unless } x = y}$

$R_1 = \{(1,1), (2,2), (3,3), (1,3)\}$ (self loop allowed)

Asymmetric Relation :-

$\forall x, y \in A$ $xRy \Rightarrow y\not{R}x$ (self loop not allowed)

Every Asymmetric is IR-Reflexive

Asymmetric always Anti-Symmetric

AS ⇒ IR

AS ⇒ ATS

★ $\boxed{IR + ATS \Leftrightarrow AS}$

★ $\boxed{S \ \& \ ATS \Leftrightarrow \text{only self loops in Relation}}$

R = {ϕ} satisfies all the Relations except the "Reflexive".

Transitive :-

$\forall x, y \in A$ $xRy \ \& \ yRz \Rightarrow xRz$

If xRy, yRz are not there so It is transitive

otherwise It has to be $xRy \ \& \ yRz \Rightarrow xRz$

eg :- $\{(1,1), (3,2), (3,3), (1,3)\}$

ignore self loops, they are transitive
so only (1,3) so transitive.

eg :-

$R_4 = \{ (x,y) \mid x+y=10 \}$ on Z (int)

$R_5 = \{ (x,y) \mid x \leq y \}$ on Z (int)

$R_6 = \{ (x,y) \mid x \parallel y \}$ on set of lines on a plane

$R_7 = \{ (x,y) \mid x \perp y \}$ on set of points on a plane

$R_8 = \{ (x,y) \mid x$ is 1 inch from $y \}$

$R_9 = \{ (x,y) \mid x$ is brother of $y \}$

$R_{10} = \{ (x,y) \mid y = x^i \; \exists \; i \in z \}$ In int it is Anti sym but in Real its not

	R	IR	S	ATS	AS	T
R_4	×	×	✓	×	×	×
R_5	✓	×	×	✓	×	✓
R_6	✓	×	✓	×	×	✓
R_7	×	✓	✓	×	×	×
R_8	×	✓	✓	×	×	×
R_9	×	✓	×	×	×	×
[illegible]	[illegible]	[illegible]	×	✓	×	✓

* $x < y$ & $y < x \Rightarrow x = y \Rightarrow$ Anti Symmetric

f so implication true

$\rightarrow (x,y) \mid y = x^{i} \exists\, i \in Z$

xRy & $yRx \Rightarrow x = y$ Anti-Symmetric

$x = y^{i_1}$

$y = x^{i_2}$

$x = y$

so $i_1, i_2 = +1$ or -1

for $i = -1$

$x = 1/y$ but in integer It doesn't have value

$y = 1/x$

so R_{10} ATS

always R, S, T

Ordered Pairs :-

eg:- I- $(x_1, y_1) R (x_2, y_2)$ Iff $x_1 + y_1 = x_2 + y_2$

If equivalence on x, y

II- $(x_1, y_1) R (x_2, y_2)$ Iff $x_1 \leq y_1$ & $x_2 \leq y_2$

both side $\leq$, and & then R, ATS, T

for Anti Symmetric

$(x_1, y_1) R (x_2, y_2)$ & $(x_2, y_2) R (x_1, y_1) \Rightarrow (x_1, y_1) = (x_2, y_2)$

	R	IR	S	ATS	AS	T	
I.	✓	X	✓	X	X	✓	Equivalence Relation
II.	✓	X	X	✓	X	✓	Partial order

→ $x_1 \le y_1$ Or $x_2 \le y_2$

$\le$ wid or only satisfies "Reflexive"

not a partial order set.

**

Counting of Relations :-

? ~~Reflexive~~

? Relation $|A| = m$ $|B| = n$ → 2^{mn}

? Relation $|A| = n$ → 2^{n^2}

? " Reflexive → 2^{n^2-n}

? " IRReflexive → 2^{n^2-n}

? " Symmetric → $2^n \times 2^{\frac{n^2-n}{2}} \Rightarrow 2^{\frac{n^2+n}{2}}$ or $2^{\frac{n(n+1)}{2}}$

? " Antisymmetric → $2^n \times 3^{\frac{n^2-n}{2}}$

? " Asymmetric → $1 \times 3^{\frac{n^2-n}{2}}$

? " - - Equivalence →

	Smallest	Largest Rel^n
$\lvert A\rvert=m$ $\lvert B\rvert=n$ Relation	$\lvert\emptyset\rvert=0$	$\lvert A\times B\rvert = mn$
Rel^n with $\lvert A\rvert=n$	$\lvert\phi\rvert=0$	$\lvert A\rvert\times\lvert A\rvert = n^2$
Reflexive	$\lvert I_a\rvert = n$	$\lvert A\times A\rvert = n^2$
IR-Reflexive	$\lvert\emptyset\rvert=0$	$\lvert A\times A - I_a\rvert = n^2-n$
Symmetric	$\lvert\phi\rvert=0$	$\lvert A\times A\rvert = n^2$
★ Anti-symmetric	$\lvert\phi\rvert=0$	$\lvert X\rvert = \frac{n(n+1)}{2}$
Asymmetric	$\lvert\phi\rvert=0$	$\lvert X\rvert = \frac{n(n-1)}{2}$
Equivalence	$\lvert I_a\rvert = n$ or $\Delta_a = n$	$\lvert A\times A\rvert = n^2$
Note :- Partial order	$\lvert I_a\rvert = n$	$\lvert X\rvert = \frac{n(n+1)}{2}$

$\lvert I_a\rvert$ = identity fn^c or Δ_a only self loops (all)

★ $R \cap IR = 0$

★ $R \cap S = 2^{\frac{n^2-n}{2}}$

★ $R \cap ATS = 3^{\frac{n^2-n}{2}}$

★ $R \cap AS = 0$

★ $IR \cap S = 2^{\frac{n^2-n}{2}}$

★ $IR \cap ATS = 3^{\frac{n^2-n}{2}}$

★ $IR \cap AS = 3^{\frac{n^2-n}{2}}$

★ $S \cap ATS = 2^n$

★ $S \cap AS = 1$

★ $ATS \cap AS = 3^{\frac{n^2-n}{2}}$

→ Every Relation has a matrix

$|A| = 4$

$$M_R \quad \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0,1 & - & - & - \\ 2 & - & 1 & 1 & - \\ 3 & - & - & - & 1 \\ 4 & - & - & - & - \end{array}$$

So $Rel^n = \underline{2^{n^2}}$

$$M_R \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

Reflexive → 2^{n^2-n}

$$\begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

IR. Reflexive

2^{n^2-n}

→ R is symmetric matrix

if $M_R^t = M_R$

transpose

$$\begin{bmatrix} 0,1 & - & - & - \\ - & 0,1 & 0,1 & 0,1 \\ - & - & - & - \end{bmatrix}$$

2^n

Symmetric → $2^{\frac{n^2+n}{2}}$ or $2^n \times 2^{\frac{n^2-n}{2}}$ or $2^{\frac{n(n+1)}{2}}$

→ Antisymmetric →

0 0 —
0 1 —
1 0 — can be taken so
1 1 X

$2^n \times 3^{\frac{n^2-n}{2}}$

Asymmetric → IR-Reflexive

$$1^n \times 3^{\frac{n^2-n}{2}}$$

$$\begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

$$2^n \times 3^{\frac{n^2-n}{2}}$$

Closures :-

Symmetric   Reflexive   Transitive

① T is Reflexive closure of R iff

② SC / 1. $R \subseteq T$

③ TC 2. T ① Reflexive / ② symmetric / ③ Transitive

3. T must be smallest such reflexive

~~T is Trans~~

eg:- $A = \{1, 2, 3\}$

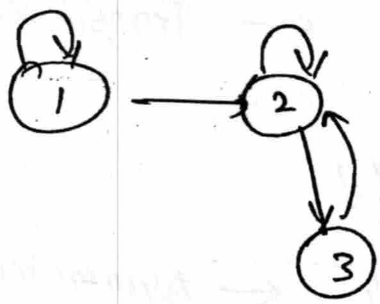
$R = \{(1,2), (2,3), (3,2), (1,1), (2,2)\}$

X $T = \{(1,2), (2,3), (3,4), (1,1), (3,2), (1,3), (2,4), (1,4)\}$

Transitive Closure

:- Other method :-

$\{ (1,2)\ (2,3)\ (3,2),\ (1,1),\ (2,2) \}$



T.C :- $\{(1,1),\ (1,2),\ (1,3),\ (2,2),\ (2,3),\ (3,2),\ (3,3)\}$

→ warshall's algo calculates transitive closure with $O(n^3)$ complexity.

→ Transitive closure of R = R^+ or R^∞ (connectivity Rel^n)

→ R^* = Reflexive Transitive closure (Reachability Rel^n)

$$\boxed{R^+ \cup I = R^*}$$

By Set - Builder method :-

$\{(x,y) \mid x<y\}$ on Z

a) $\{(x,y) / x<y\}$ ← Transitive closure

b) $\{(x,y) \mid x=y\}$

c) $\{(x,y) \mid x \neq y\}$ ← symmetric

d) $\{(x,y) \mid x \leq y\}$ ←

Reflexive closure :- $\{(x,y) / x \leq y\}$

Symmetric closure :-

→ R is symmetric iff $R^{-1} = R$

$\{x,y / x<y\} \cup \{x,y / y<x\}$

$\{x,y \mid x<y \text{ or } y<x\}$

$\{x,y \mid x \neq y\}$

Que:- Transitive closure :-

$\{(x,y) \mid y=x+1\}$ on Z

$y = x+1$

$z = y+1$

$z = x+x+1 = x+2$ ← Not transitive

make it :-

on a set of n elements

$\{1, 2, \ldots\}$

TC :- $R \cup R^2 \cup R^3 \ldots R^n$
or
$R \cup R^2 \cup R^3 \ldots R^{\infty}$ same

but for ∞ elements

$R \cup R^2 \cup R^3 \ldots R^{\infty}$

$R^2 = R(R(x))$
$R(x+1)$
$R^2 = x+2$

$\Rightarrow \{(x,y) \mid y = x+1\} \cup \{y = x+2\} \cup \{y = x+3\} \ldots$
$R \cup R^2 \cup R^3 \cup \ldots$

so

$Tc = \{(x,y) \mid y = x+1\} \cup \{(x,y) \mid y > x+1\}$

$\Rightarrow \{(x,y) \mid y \geq x+1\} \Leftarrow T.C.$

symmetric :-

$R = \{(x,y) \mid y = x+1\} \cup \{(x,y) \mid x = y+1\}$

$= \{x,y \mid y = x+1 \text{ or } \frac{x = y+1}{y = x-1}\}$

$= \{(x,y) \mid y = x+1 \text{ or } x-1\}$

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Equivalence Relation :-

- equivalence, Quoitent set (Partition) :

$A = \{1, 2, 3, 4\}$

$R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4)\}$

R is equivalence Rel^n

then Quoitent set = ?
A/R (π)

π is a Partition of A :-

1) $|A_i| \geq 1$ (i.e. It should not contain empty Block) Non empty

2) $i \neq j \Rightarrow A_i \cap A_j = \phi$ (Blocks must ~~not~~ have nothing in common) (mutually exclusive)

3) $\cup A_i = A$

a) $\{\{\}, \{1, 2, 3, 4\}\}$ 1st cond. ✗

b) $\{\{1, 2\}, \{2, 3, 4\}\}$ 2nd cond ✗

c) $\{\{1, 2\}, \{3\}\}$ 3rd cond ✗

d) $\{\{1\}, \{2, 3, 4\}\}$ ✓

:- Quotient Set

A/R = Set of Distinct equivalence classes of 'every element' of A.

A {1, 2, 3, 4}

{1} - -

{2} - -

{3} - -

{4} - -

$R = \{(1,1), (1,2), (2,2), (2,1), (3,3), (3,4), (4,3), (4,4)\}$

equivalence class :- $\forall x \in A$

(x)

$[x] = R(x)$

$[1] = \{1, 2\}$

$[2] = \{1, 2\}$

$[3] = \{3, 4\}$

$[4] = \{3, 4\}$

$A/R (\pi) = \{\{1, 2\}, \{3, 4\}\}$

Property of equivalence class :-

1) $|E_i| \geq 1$

2) $i \neq j \Rightarrow E_i \cap E_j \neq 0$

3) $\cup E_i = A$

same as Partition Property.

→ Every equivalence class is a partition, If R is a equivalence rel^n of R.

→ every equivalence Rel^n is a partition of R.