Date
03/09/2017

Poset - LATTICES & BA

1. Poset, Toset & woset : → well ordered set
2. Topological Sorting
3. Hasse Diagram
4. Extremal Element of Poset : Max, Min, G, L, UB, LB, LUB, GLB
5. Lattice
6. Properties of Lattice
7. Types of Lattice : Bounded Lattice, complimented Lattice, Distributive Lattice., Semi - Lattice
8. Boolean Algebra

:- "Poset" (Partially ordered set) :- A Non Empty set along with partial order relation. $(S, \leq)$

Poset :- (Reflexive, Antisymmetric, Transitive.) :- (R, ATS, T)

:- Simplest Poset $(S, \leq)$

:- Standard Posets :-

1) $(Z, \leq)$ $(R \leq)$ Infinite Poset

2) $(\{1,2,3,4\}, \leq)$ finite Poset

eg :- $(S, \subseteq)$ $\longrightarrow$ subset also always form a Poset.

:- Power set of a given set is always a Poset.

becoz Powerset of even empty set is not empty.

eg: $S = \{1, 2\}$

$(P(S), \subseteq)$

$P(S) = \{\phi, \{1\}, \{2\}, \{1, 2\}, \subseteq\}$ finite Poset.

$(Z, /) \rightarrow$ divide

eg:- xRy iff x/y $\longrightarrow$ Its not a Poset because Its not Reflexive & Antisymmetric

$y = kx$

(becoze of 0)

$\underbrace{\frac{2}{-2} \ \& \ \frac{-2}{2}}$ symmetric but not antisymmetric

:- but $(Z, /)$ is Transitive.

But $(Z^+, /)$ is poset

because It's not have, zero & Neg-Integers.

Its a infinite set.

$(D_n, /)$ finite set

D_n : Set of all positive integers divisor of n

$D_6 : \{1, 2, 3, 6\}$ $\quad D_{20} = \{1, 2, 4, 5, 10, 20\}$

$D_{10} = \{1, 2, 5, 10\}$

of Divisors :-

$D_{20} = ?$

Break down into Prime factors

$$\begin{array}{r|l} 2 & 20 \\ \hline 2 & 10 \\ \hline & 5 \end{array}$$

$$20 = 2^2 \times 5^1 = P_1^{n_1} \times P_2^{n_2} —$$

$$|D_n| = (n_1+1)(n_2+1) \cdots (n_r+1)$$

$$= (2+1)(1+1)$$

$$= 3 \times 2 = \underline{6}$$

:- $(\bar{Z}, /)$ also a Po-set as It also not contains zero & +ve integers.

eg:- $x \leq y$ & $y = x^2$ It also Poset.

(y is integral power of x)

eg:- :- Product Partial order set :-

:- $(x_1, y_1) \; R \; (x_2, y_2)$ iff $x_1 \leq x_2$ & $y_1 \leq y_2$

$(Z \times Z, \leq)$ Poset

$(R \times R, \leq)$ Poset

:- $(x_1, y_1) R (x_2, y_2)$ iff $x_1 \le x_2$ or $y_1 \le y_2$

↓

Not Transitive so Not a Poset.

(It also not be Anti-symmetric)

:- To-Set :- (Linearly ordered set) (Totally ordered set)

Poset + comparability = Toset

$(S, \le)$

↓

$\forall x, y \in S$ $\boxed{x \le y \text{ or } y \le x}$

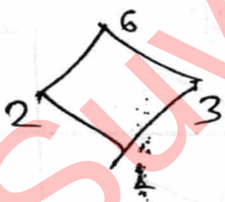
	Poset	Toset (chain)	woset
$(Z, \le)$, $(R, \le)$, $(N, \le)$	✓	✓	only $(N, \le)$ & $(Z^+, \le)$ ✓
$(C, \le)$	✓	✗	✓
$(\{1,2,3,4\}, \le)$	✓	✓	
$(S, \subseteq)$	✓	✗	
$(P(S), \subseteq)$	✓	✗ only if $\lvert S\rvert = 0$ or 1	
$(Z^+, /)$	✓	✗ ~~if $\lvert S\rvert = 0$ or 1~~	
$(D_n, /)$	✓	✗ D_8, P_{16}, D_P ✓	If Toset then woset
$(\{1, 3, 10, 16\}, /)$	✓	case by case ✗	〃
$\{Z \times Z, PPO\}$	✓	✗	
$\{R \times R, PPO\}$	✓	✗	

:- Powerset with more than 1 element will not be a Toset because Hasse diagram will not be chain.

:- And Poset is Toset if the Hasse diagram is a chain.

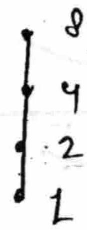
$\{D_n, |\}$:- some of $\{D_n, |\}$ are Toset & some are not

eg:- $D_6 = \{1, 2, 4, 6\}$ is not Toset



Not a chain

$D_8 = \{1, 2, 4, 8\}$



Its a chain

$\{D_8, |\}$ is Toset.

$\{Z \times Z, PPO\}$ not a Toset

$(2,3) \leq (4,2)$

as $2 < 4$ & $3 > 2$ so incomparable

:- wo-set (well ordered set) :-

Toset + least element = woset
$\Downarrow$
Discrete set

Note

:- If R is a poset then R^{-1} is also a Poset i.e. Every Poset has a dual Poset.

But dual of woset ~~is~~ may not be a woset

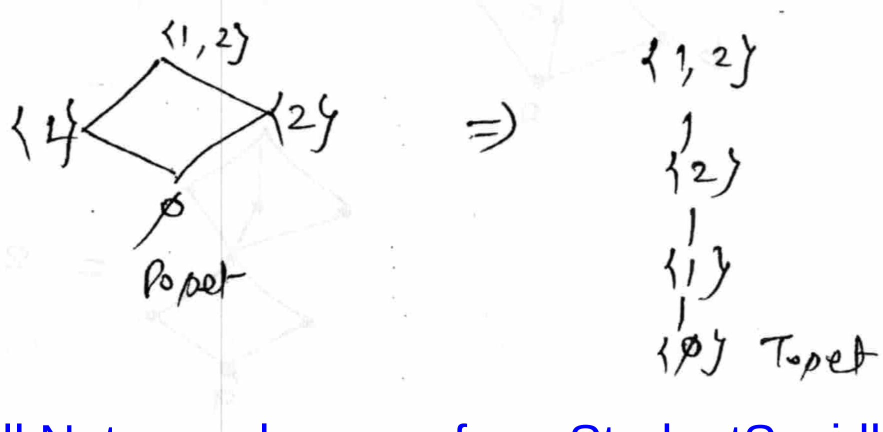
$(Z^{+}, \leq) \rightarrow$ woset

$(Z^{-}, \geq) \rightarrow$ is also a woset because Its also have least element's chain.

:- If D_n is a Toset then It also be woset.

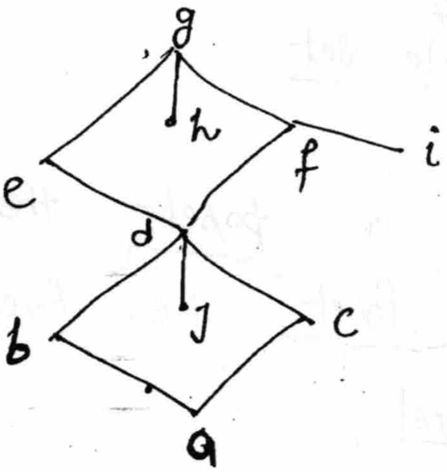
(2) :- "Topological Sorting" :-

:→ Converting a Poset to compatible Toset.



Convert the Poset (Hasse diagram) to chain without violating the given order

eg :-



Algorithm :-

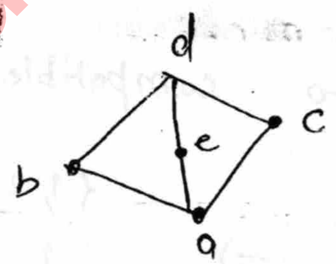
Step → 1st Select any minimal element (which not having any incoming arrow)

:– By default Lattice have all up going arrows

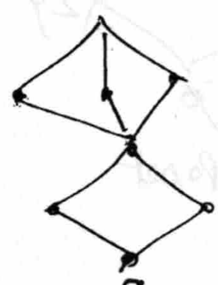
so minimal elements (a, J, h, i)

Step – 2nd :- Delete minimal element and goback to step - 1st.

:– # of Topological Sort :—



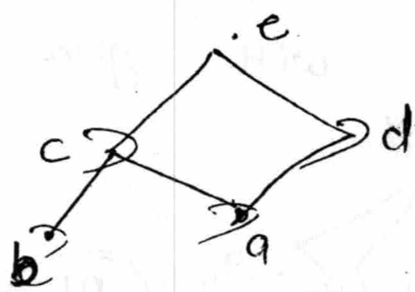
= 3! = 6



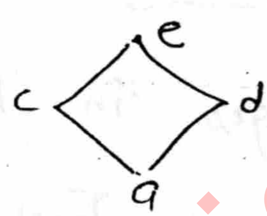
= 2! × 3! ⇒ 12

If diagram asymmetric, don't use factorial method.

eg:



so It can be start with a or b

$b \le$ 

$\Rightarrow 2! = 2$

$a \le b$ 

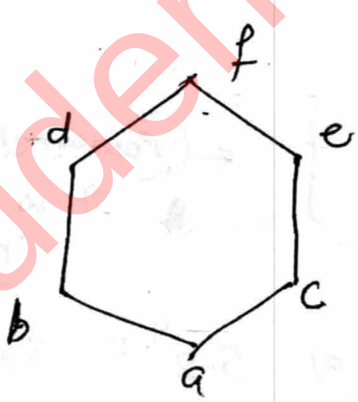
$\Rightarrow 2! = 2$

$a \le d$ 

$= 1$

so 5 topological sorts

eg:



$a \le b \le c \Rightarrow 2!$ $\Rightarrow 3!$

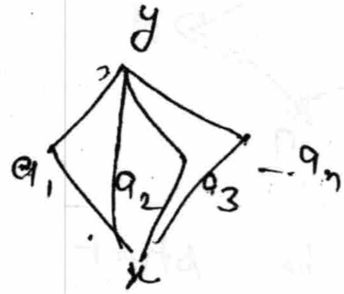
$a \le b \le d \Rightarrow 1$

$a \le c \le b = 2!$ $\Rightarrow 3!$ $\Rightarrow \underline{6}$

$a \le c \le e = 1!$

Que:- Consider a Poset $x \le a_i$ $\forall i = 1, 2, 3, \ldots$; $a_i < y$ $\forall i = 1, 2, 3 \ldots$

How many Topo sort with given Poset



n!

3.) :- Hasse Diagram :-

~~#~~ # of edges in Hasse diagram ?

eg: If a Poset is a Toset then

$(\{1, 2, 5, 10, 16, 23\}, \le) \implies$

# of edes = 6 − 1 = 5

No. of element − 1 = 5

eg:

(Powerset (S) , ⊆)

$\boxed{\#\ \text{edges}\,(e) = n \cdot 2^{n-1}}$ ← (Powerset, ⊆) It is called 'n-cube'

(P(5), ⊆) $5 \cdot 2^4 = 5 \times 16 = 80$

Note ★ n-cube graph: no. of edges $= n \cdot 2^{n-1}$

no. of virtices $= 2^n$

Q_2 :- 2 cube

Q_3 :- 3 cube

eg:- $(P(S), \subseteq)$

$|P(S)| = 32$ i.e. $2^n = 32$

↳ cardinality of powerset.

so $\log_2 32 = \underline{5}$

$5 \cdot 2^4 \Rightarrow \underline{80}$

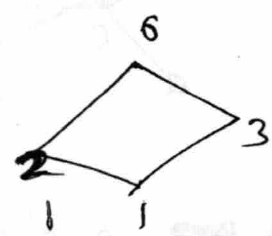
eg:-

$|P(S)| = n$

$$\# e = \log_2 n \cdot 2^{\log_2 n - 1}$$

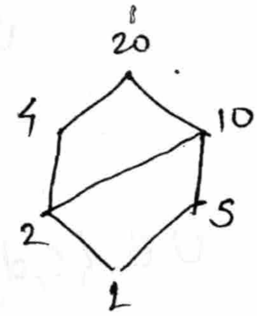
:- $(D_n, |)$

eg $(D_6, |)$ $D_6 = \{1, 2, 3, 6\}$



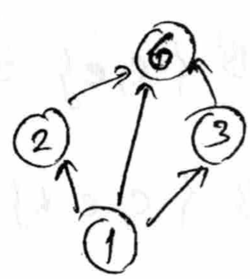
$D_{10} = \{1, 2, 5, 10\}$

$D_{20} = \{1, 2, 4, 5, 10, 20\}$

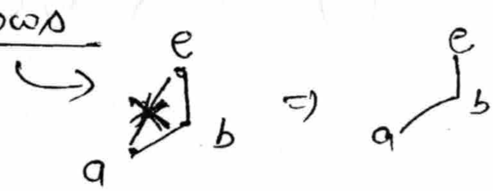


Digraph → Hasse diagram

1. Remove self loops

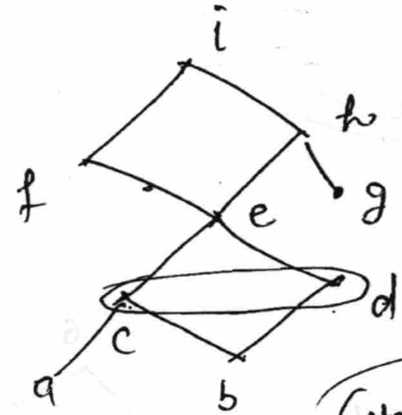


2. Draw it such that all the arrow going up & remove arrow head.
3. Remove "Transitive arrows"



:- Every Poset has unique ~~greater~~ greatest element (False)
becoz (because) It may not exist

Bounded Poset :- If Poset have both G & L then It is called Bounded.



minimal = {a, b, g} (LX)
maximal = {i} = G

(Not Bounded)

$A \subseteq S$

$UB(A)$: $a \in S$ If $UB(A)$ Iff $\forall_b \in A$ $b \leq a$

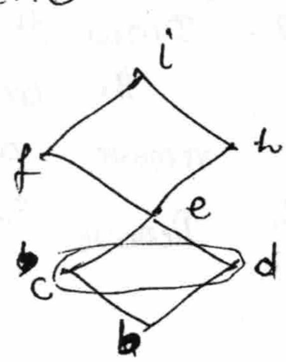
$LB(A)$: $a \in S$ is $LB(A)$ Iff $\forall_b \in A$ $a \leq b$

eg $UB\{c,d\} = \{e, f, h, i\}$, c,d incomparable

$UB\{c,e\} = \{e, f, h, i\}$ e>c so e in UB.

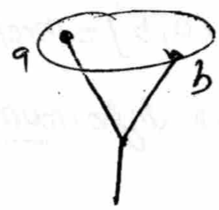
$LB\{c, e, d\} = \{c, b\}$

$LB\{h\} = \{e, c, d, b\}$

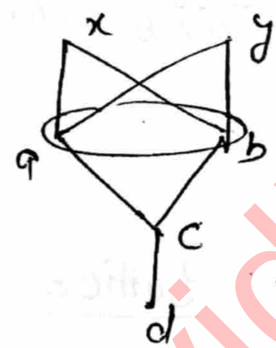


★★ LUB (unique): $a \in LUB(A)$ iff
1. $a \in UB(A)$
2. $\forall c \in UB(A)$ $a \leq c$

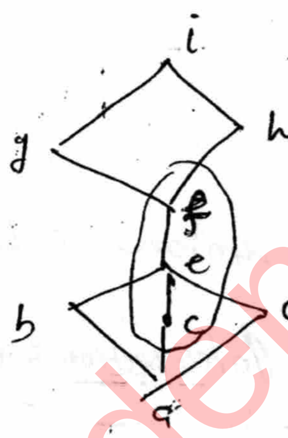
★★ GLB (unique)
greatest lower Bound
→ $a \in GLB(A)$ iff
1. $a \in LB(A)$
2. $\forall c \in LB(A)$ $c \leq a$



No UB
so no LUB



x, y incomparable
so No LUB



all are comparable
LUB (f e c) = {f}
GLB (f e c) = {c}

If $x \leq y$ iff $x \vee y = y$
& $x \wedge y = x$

"Lattice" :-

A Po-set $(S, \leq)$ is a lattice iff $\forall a, b \in S$ Both LUB$\{a,b\}$ & GLB$\{a,b\}$ must exist & belong to L.

:- LUB $\{a,b\} = a \vee b = a+b = \text{Sup}\{a,b\} = \text{Join}\{a,b\}$

Sup → Supremum

:- GLB $\{a,b\} = a \wedge b = a \cdot b = \text{Inf}\{a,b\} = \text{meet}\{a,b\}$

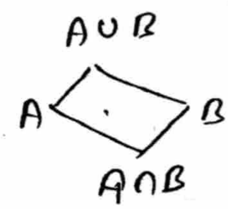
Inf → Infimum.

Standard Lattice :-

	Lattice	
$(Z, \leq)$ $(R, \leq)$	✓	
$(\{1,2,3,4\}, \leq)$	✓	
$(S, \subseteq)$	✓	union = LUB
$(P(S), \subseteq)$	✓	Intersection = GLB
$(Z^+, \mid)$	✓	→ LCM & GCD
$(D_n, \mid)$	✓	→ LCM & GCD
$(\{1,3,5,10\}, \mid)$	may or may not ✓✗	
$(\{Z \times Z\}, PPO)$	✓	LUB $(\max(x_1, x_2), \max(y_1, y_2))$
$(\{R \times R\}, PPO)$	✓	GLB $(\min(x_1, x_2), \min(y_1, y_2))$

:- Every To-set always be a lattice.
But reverse is not true. i.e.
A Lattice may not be a To-set.

:- In Subset operation LUB always be union.



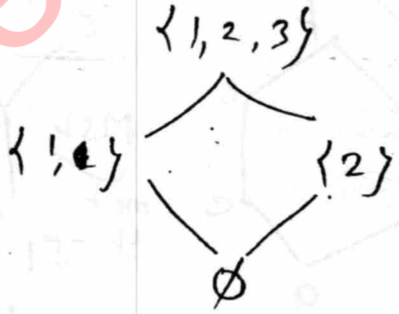
:- & In Subset opⁿ GLB always be intersection

:- In Powerset also same, LUB ⇒ (A∪B)
GLB ⇒ (A∩B)

eg:- $S = \{ \phi, \{1\}, \{2\}, \{1,2,3\} \}$

$(S, \subseteq)$ is a lattice but

$a \cup b$ is not LUB

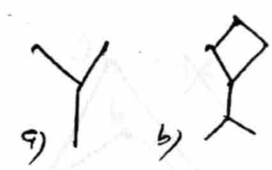


$a \cup b \neq$ LUB
$\{1\} \cup \{2\} = \{1,2\}$

:-

~~check only in~~

Just two cases:- 1)



Not a Lattice 2)

as LUB or GLB not exist.

eg :- $S = \{1, 2, 3, 4\}$

$\{1,2\} + \{3,4\} \cdot \{4,2\}$

$\{1,2\} + \{4\}$

$= \{1,2,4\}$

If set the

$+ = \cup$

$\cdot = \cap$

eg :- $2 + 5 \cdot 7$

$2 + 1$

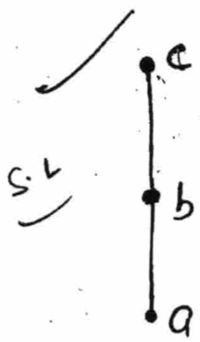
$\{2\}$

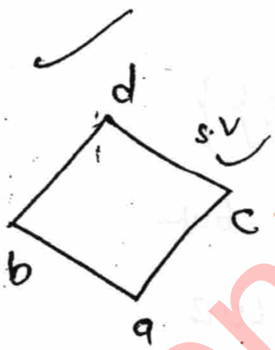
If not a set

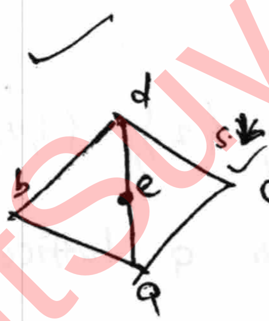
$+ =$ LCM

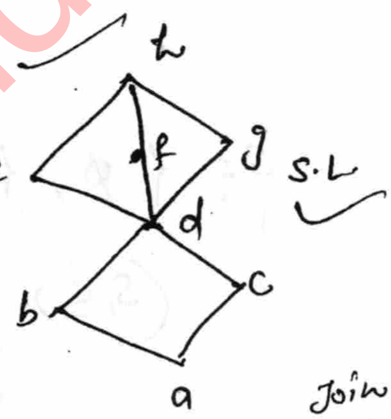
$\cdot =$ GCD

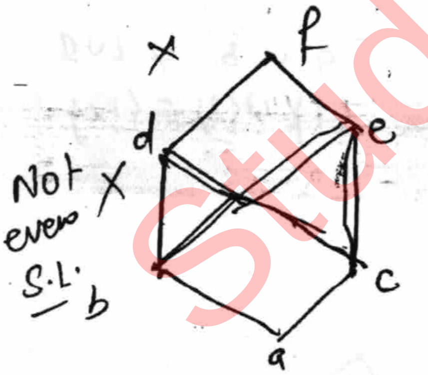
Lattice :-

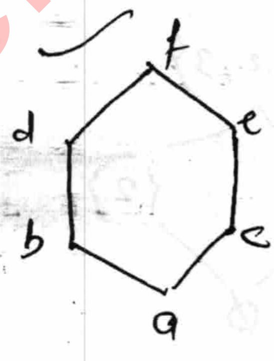


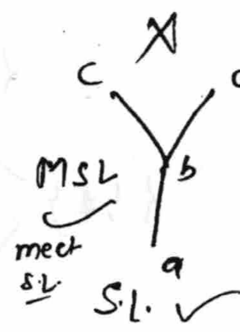


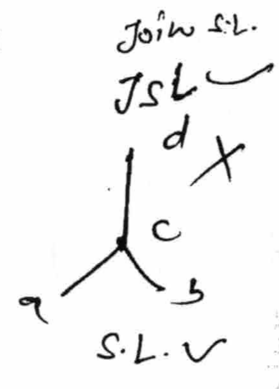


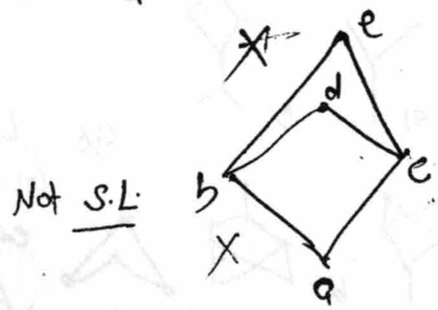


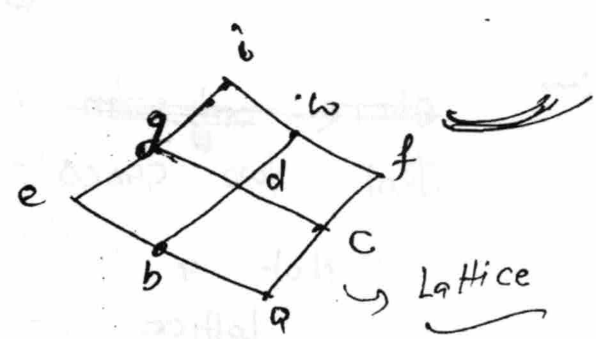












Semi-Lattice :-

A Poset $(S, \leq)$ is sublattice iff

$\forall a,b \in S$ (LUB $\{a,b\}$ or GLB $\{a,b\}$) ✗

because $\forall$ not distributive over 'OR'

A Poset $(S, \leq)$ is sublattic iff

$(\forall a, b \; LUB(a,b))$ or $(\forall a, b \in S \; GLB(a,b))$

:- Every lattice is both JSL & MSL (Join sub lattice & meet sub-lattice).

Properties of Lattice :-

✓ 1. Closure
✓ 2. Commutative
✓ 3. Associative

necessory

Derived

✓ 1. IDempotent
✓ 2. Law of Absorption

necessory

✗ 4. Distributive
✗ 5. Identity
✗ 6. Complement

✗ 3. Double compliment
✗ 4. De-morgan's
✗ 5. Domination

"Law of Absorption" :-

$$a \vee (a \wedge b) = a$$
$$a \wedge (a \vee b) = a$$

i) $a \leq a \vee b \qquad a \wedge b \leq a$ "Consistency"
$b \leq a \vee b \qquad a \wedge b \leq b$

ii) $\boxed{a \leq b \ \text{ iff } \ a \vee b = b \ \& \ a \wedge b = a}$

In lattice these properties also exist

iii) $a \leq b$
$\& \ c \leq d \Rightarrow a \vee c \leq b \vee d$
$a \wedge c \leq b \wedge d$
(one way)

iv) **Distributive Inequality** :-

$$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$$
$$(a \wedge b) \vee (a \wedge c) \leq a \vee (b \wedge c)$$

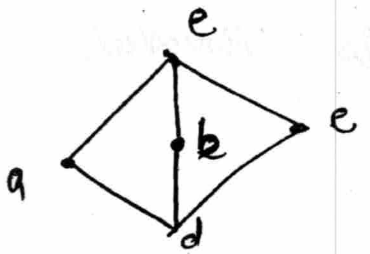
eg:- which ~~one~~ of the following are true?

I. $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$ ✓

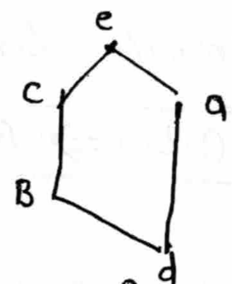
II $(a \vee b) \wedge (a \vee c) \leq a \vee (b \wedge c)$ ✗

III $a \vee (b \wedge c) \leq (a \wedge b) \vee (a \wedge c)$ ✗

IV. $(a \wedge b) \vee (a \wedge c) \leq a \vee (b \wedge c)$ ✓



Kite Lattice



Pentagon Lattice

Types of Lattice :-

1) **Bounded Lattice :-**

:- A Lattice is bounded $(L, \leq)$ iff 0, 1 (i.e. Greatest, least) element $\in L$

:- Every finite lattice is bounded but Bounded lattice may not be finite.

$$FL \Rightarrow B$$

:- $(1 \leq x \leq 10,\ x \in \text{Real No})$ → Infinite Bounded Lattice

:- $(S, \subseteq)$ not Bounded

:- $(P(S), \subseteq)$ is Bounded

:- $(D_n, /)$ Bounded

2) **Compliment Lattice :-**

A lattice is C.L. $(CL, \leq)$ iff $\forall a\ \exists a'$

$$a + a' = 1$$

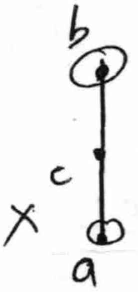
& $a \cdot a' = 0$

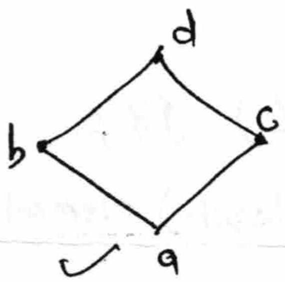
:- Every compliment Lattice is Bounded.

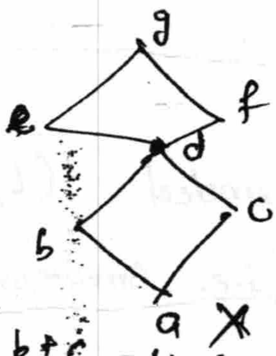
$$C.L. \Rightarrow B.L$$

$0'$ (Least) $= 1$

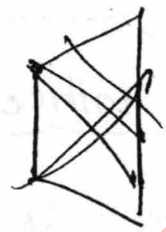
$1'$ (greatest) $= 0$

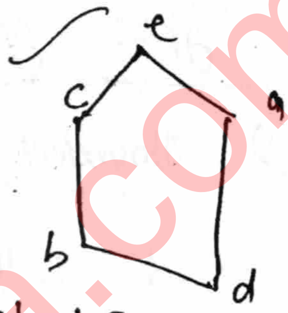






$b + c = !$LUB





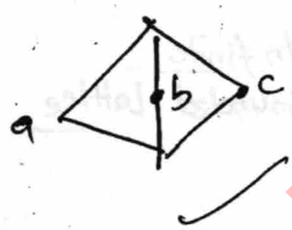
$a' = bc$

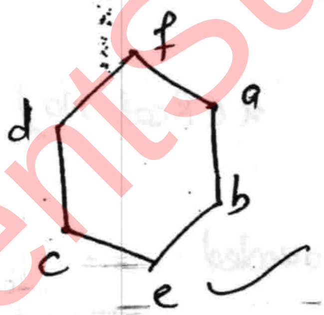
$b' = a$ $\quad c' = a$

:- compliment will not be comparable so check only incomparable & also check b/w them

$a + a' = 1$ (LUB)

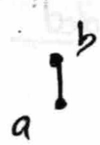
& $a \cdot a' = 0$ (GLB)

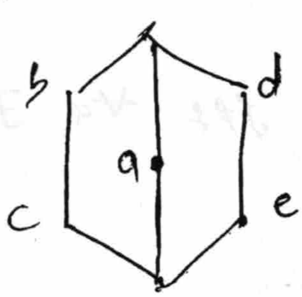




:- Toset is complimented lattice only Iff

$n = 2$





$a' = 4$ compliments $\{b, c, d, e\}$

$b' = 3$ $\{a, d, e\}$

$c' = 3$ $= \{a, d, e\}$

$d' = 3$ $e' = 3$

:- Distributive Lattice :-

Distributive Lattice satisfy distributive property i.e. Distributive equality as well as enequality.

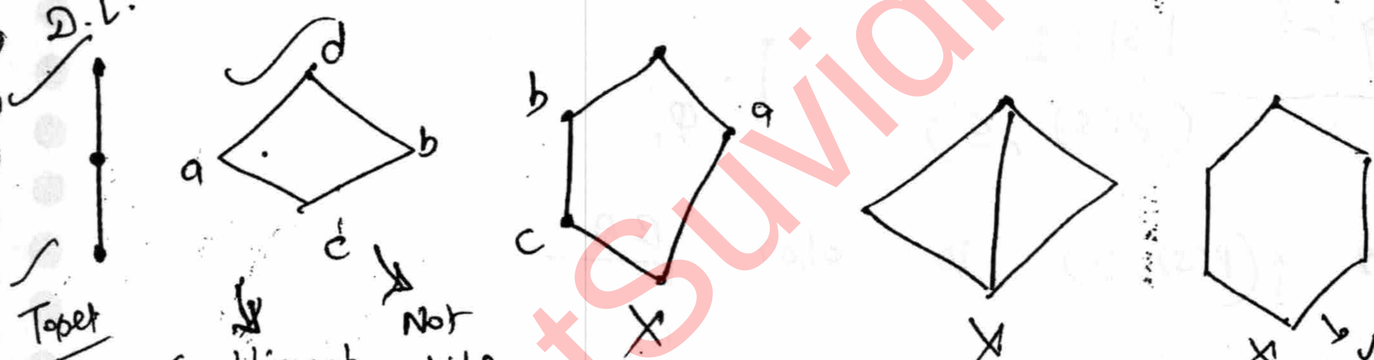
i.e

$$\forall a, b, c \in L \quad \boxed{\begin{array}{l} a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \\ \& \ a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \end{array}}$$

1. $\boxed{\text{C.L. is Distributive} \Rightarrow \text{c is unique compliment}}$ (one way)

:- If compliment not unique then It is not Distributive

D.L.



Toset

Compliment unique. (Distributive may or may not)

Not kite Lattice

It has Pentagon Lattice also

Compliments are not unique not Distributive lattice.

2. Lattice $(L, \leq)$ is D.L. Iff It does not contain kite Lattice or Pentagon lattice as sublattice.

→ Every Toset is always Distributive lattice.

Boolean algebra :-

1. $(D_n, /)$ is a B.A iff $n = \underbrace{p_1^{n}, p_2^{n} --}_{\text{No Repetition}}$

Boolean algebra is Bounded, Compliment & Distributive . Lattice.

(or) Boolean algebra is complimented & Distributive Lattice

C.L. & D.L. ≡ B.A.

eg :- $|S| = 1$

$(P(S), \subseteq)$ a_1

2. $(P(S), \subseteq)$ is also B.A

eg:- D_{30}, D_6, D_{10}, D_p are B.A.

:- Properties :-

2. $\boxed{a \leq b \text{ iff } ab' = 0}$

8. $a = b$ iff $ab' = 0$ & $a'b = 0$.

3. Left cancellation & Right cancellation do not hold.

$ab = ac \Rightarrow b = c$ ✗

$b = c \Rightarrow ab = ac$ ✓