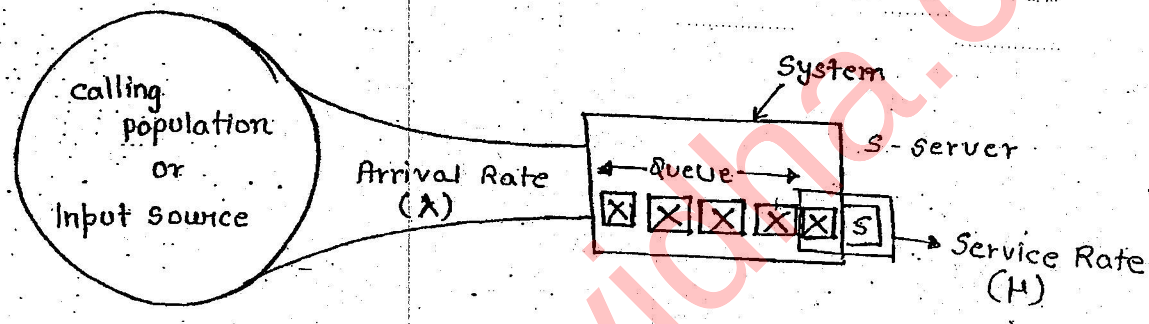
Queing Theory and Waiting Line Theory

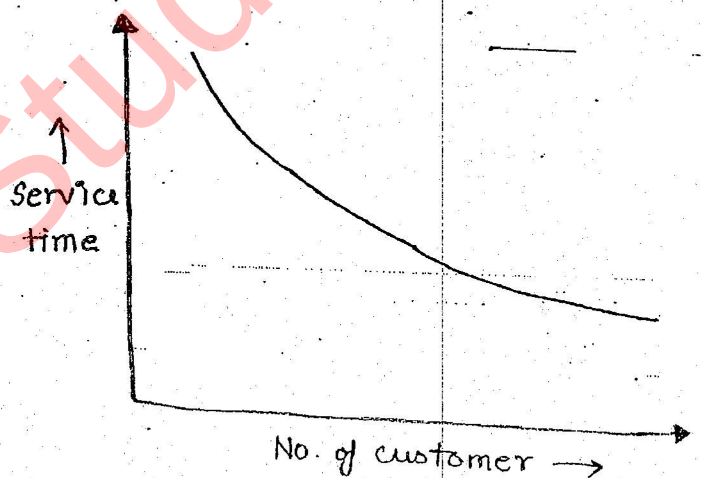
The aim of queing theory is achievement of economic balance b/w the cost of providing service and the cost associated with the weight required for that service. It is applied to service ~~based~~ oriented organisation like production shop, work shop, repair shop, food chain, bank, ATM etc.

Characteristic of Queing model :-



Arrival Rate / pattern :- The number of customer arriving per unit time is termed as arrival rate. Customer arrival is random and therefore it is assumed to follow poison distribution

Service Rate / Pattern :- The number of customer service per unit time is known as service rate and it is assumed to follow exponential distribution



Service Rule or Service order

1) FIFO or FCFS → ATM
2) LIFO or ~~LCFS~~ → Loading & unloading
3) SIRO → strike
4) Priority treatment → VIP

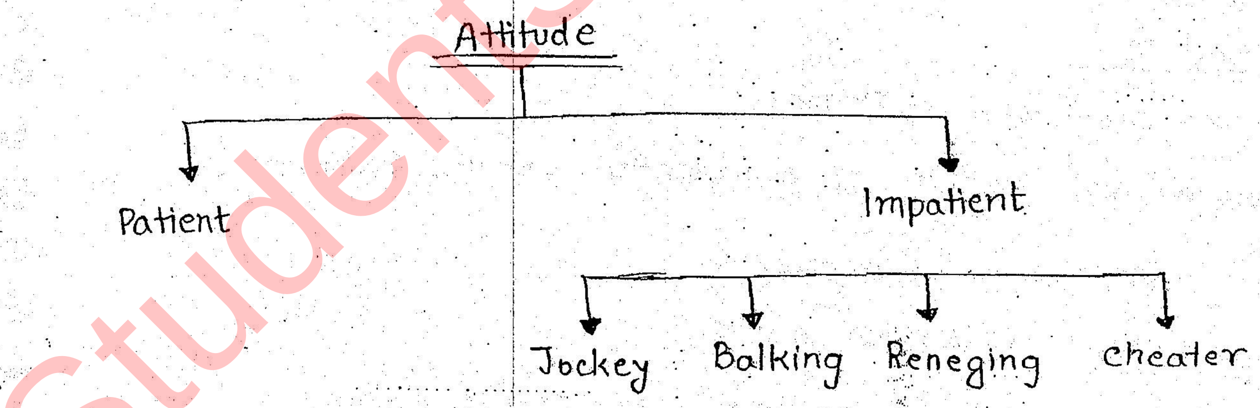
Service rule give information about queue discipline which means the order by which customer are picked from waiting line in order to provide service.

System and Calling Population :-

System is the place or facility where customer arrive in order to get service and its capacity may be finite or infinite.

The entire sample of customer from which few visit the system is termed as calling population or input source. Its capacity may be finite or infinite. It is infinite when the arrival of few customer does not have any effect on the arrival pattern of future customer.

Customers Attitude :-



Jockey :- Customer keep on changing queue in hope to get service faster.

Balking :- Customer does not join the Queue and leave the system as queue is very long.

Reneging :- Customer join the queue for short duration and then leave the system as queue is moving very slow.

Cheater :- customer take illegal mean like fighting, bribing etc., in hope to get service faster.

Representation of Queing model -

Queing model are represented by Kendall and Lee notation whose general form is -

$$\boxed{(a/b/c) : (d/e/f)}$$

where,

a → prob. distribution for arrival pattern

b → prob. distribution for service pattern

c → no. of server within the system

d → service rule or service order

e → size or capacity of system

f → size or capacity of calling population or Input source

Symbols :-

a and b

- M → Markovian (Poisson) for arrival pattern or exponential service pattern
- E → Erlangian (Gamma) for arrival or service pattern
- D → Deterministic arrival or service pattern

c → 1, 2, 3, 4, 5

d → FIFO, LIFO, SIRO, GD → General Service Discipline

e and f

N → Finite

∞ → Infinite

Arrival Rate = λ (Poisson)

λ = 15 cust/hr

Inter-arrival rate or Time = $\frac{1}{\lambda}$ = $\frac{1}{15}$ hr/cust = 4 min/cust

(Exponential)

Service Rate = μ (Exponential)

μ = 20 cust/hr

Inter-service rate or Time = $\frac{1}{\mu}$ = $\frac{1}{20}$ hr/cust = 3 min/cust

(Poisson)

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i) if $\lambda > \mu$ - Queue length keeps on increasing and after certain period of time incomming population will not get service. In this condition we cannot find sol^n as system ultimately fails.

ii) if $\lambda < \mu$ - $\lambda \leq \mu \rightarrow$ system work

$\lambda < \mu \rightarrow$ prefer

The ratio of Arrival to service rate indicates the ~~or~~ percentage time the server is busy as is known as utilisation factor, average utilisation, system utilisation, channel efficiency and clearing ratio. It also indicate the probability that the new customer have to wait.

$$\boxed{\rho = \frac{\text{Arrival Rate}}{\text{Service Rate}}}$$

a) Probability that the system is idle or probability of zero customer in the system

$$\boxed{P_0 = 1 - \rho}$$

b) Probability of having exactly 'n' customer in the system

$$\boxed{P_n = \rho^n P_0}$$

eg - n = 4 cust.

$$P_4 = \rho^4 P_0$$

$$P_0 + P_1 + P_2 + P_3 + \ldots\ldots = 1$$

$$P_3 + P_4 + \ldots = 1 - (P_0 + P_1 + P_2)$$

$$= 1 - (P_0 + \rho^1 P_0 + \rho^2 P_0)$$

c) Average no. of customer in the system –

In this we include both the customer waiting in the Queue along with those getting service.

$$\boxed{L_s = \sum_{n=0}^{\infty} n P_n} \longrightarrow \boxed{L_s = \frac{\rho}{1-\rho}} \qquad \rho = \frac{\lambda}{\mu}$$

d) Average no. of customer in the Queue –

In this we do not include the customer getting service

$$\boxed{L_q = \sum_{n=2}^{\infty} (n-1) P_n}$$

$$\boxed{L_q = \frac{\rho^2}{1-\rho} = L_s - \rho = L_s \cdot \rho}$$

Little's Law :– For a stable system average no. of customer in the system or queue is given by average customer arrival rate multiply by avg. ~~waiting~~ waiting time of the customer in the system or queue.

$$\boxed{L_s = \lambda W_s} \qquad W_s = \frac{L_s}{\lambda}$$

$$\boxed{L_q = \lambda W_q} \qquad W_q = \frac{L_q}{\lambda} = W_s - \frac{1}{\mu}$$

W_s → avg. waiting time of cust. in the system

W_q → " " " " " " queue.

Average $\longrightarrow$ System } If not specified.
Mean $\longrightarrow$ Queue }

Q - The no. of persons arriving at service centre is 8 cust / hr and the service provider takes 5 min / cust on an avg. Then determine -

i) L_s and L_q

ii) W_s and W_q

$\rightarrow$ $\lambda \rightarrow$ 8 cust/hr , $\cancel{\lambda = \frac{60}{8} = 7.5}$ min/cust.

$\mu = 5$ min / cust

$\cancel{\rho = \frac{\lambda}{\mu} = \frac{7.5}{5} = 1.5}$

$\cancel{L_s = 1.5}$

$\frac{1}{\mu} = 5$ min / cust

$\mu = 12$ cust / hr

$$\rho = \frac{8}{12} = \frac{2}{3} = 0.66$$

$$L_s = \frac{0.66}{0.34} = 2$$

$$L_q = \frac{4}{3}$$

2) $W_s = 15$, $W_q = 10$ min.

Q - Shopkeeper service 10 cust/ hr and the cust arrival is 8 cust/hr. Find the probability that at least 2 cust waiting in the queue

$\rightarrow$ $\rho = \frac{\lambda}{\mu} = \frac{8}{10} = 0.8$

Atleast 2 in the Queue means at least 3 in the system

$P_0 + P_1 + P_2 + \ldots = 1$

$\cancel{P_2} + P_3 + P_4 \ldots = 1 - (P_0 + P_1 + P_2)$

Atleast 'n' customer in the system = ρ^n

$= 1 - (0.2 + \rho^1 P_0 + \rho^2 P_0)$

$= 0.512$

74, Q13 → m/c → $\lambda = 6$ mc/hr

Idle mc rate = Rs 15 / hr mc

for A :- $\frac{1}{\mu_A} = 6$ min/mc , $\mu_A = 10$ mc/hr

$\rho_A = \frac{6}{10} = 0.6$ $(Salary)_A$ = Rs 8/hr

for B :- $\frac{1}{\mu_B} = 5$ min/mc , $\mu_B = 12$ mc/hr

$\rho_B = \frac{6}{12} = 0.5$, $(Salary)_B$ = Rs 10/hr

Total cost = cost of Idle mc + salary

for A → $(L_s)_A = \frac{\rho_A}{1-\rho_A} = 1.5$ m/c

$(\text{cost of Idle mc})_A = (L_s)_A \times$ Idle mc Rate × shift duration

$= 1.5 \text{ mc} \times \frac{\text{Rs } 15}{\text{hr mc}} \times 8 \text{ hr} = \text{Rs } 180$

$(\text{salary})_A$ = Rs 8/hr × 8 = Rs 64

$\therefore (T.C)_A = 180 + 64 = \text{Rs } 244$

for B → $(L_s)_B = \frac{\rho_B}{1-\rho_B} = 1$ m/c

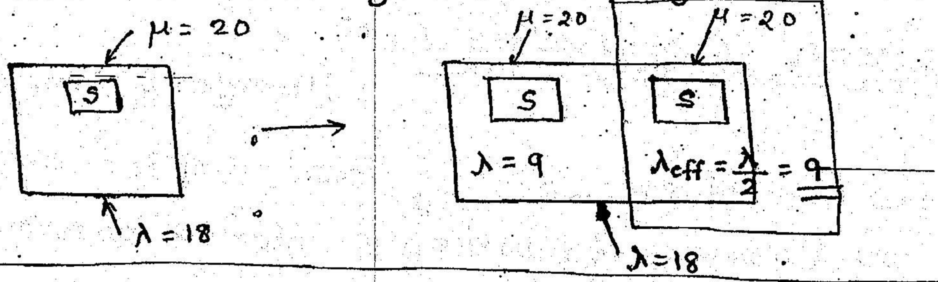
$(\text{cost of Idle mc})_B = 1 \text{ mc} \times \frac{\text{Rs } 15}{\text{hr mc}} \times 8 \text{ hr} = \text{Rs } 120$

$(\text{Salary})_B$ = Rs 10/hr × 8 = Rs 80

$\therefore (T.C)_A = 120 + 80 = \text{Rs } 200$

B should be preferred.

If 2 worker are working simultaneously -



- Average length of non empty queue or avg. length of Queue containing at least 1 customer.

$$\boxed{L_q' = \frac{1}{1-\rho}}$$

- Probability of 'n' arrival in the system during period T

$T = 15$ min
$n = 4$ cust

$$\boxed{P(n,T) = \frac{(\exp)^{-\lambda T} (\lambda T)^n}{n!}}$$

$\lambda \rightarrow$ cust / hr
$T \rightarrow$ hr

- Probability that more than 'T' time period is required to service a customer.

$$\boxed{P(T) = (\exp)^{-\mu T}}$$

- Probability that the waiting time in the queue is greater than 'T'.

$$\boxed{P(W_q > T) = \rho (\exp)^{-\frac{T}{W_s}}}$$

- Probability that ~~the~~ waiting time in the system is greater than 'T'

$$\boxed{P(W_s > T) = (\exp)^{-\frac{T}{W_s}}}$$

prob will increase.