Linear Programming (LP)

(George B. Dantzing)

LP is used for optimization of our limited resources when there are number of alternate solution possible for the problem. It is a mathematical technique and the term linear is used for variable and it simply means that the relationship b/w diff variable can be represented in the form of straight line.

Requirement of LP :-

1) **Objective Function** :- It is ~~the~~ main function which we need to optimise and it should be clearly identifiable and measurable in quantitative term like maximization of profit or sale or minimisation of cost.

2) **Constraint or condition** :- These are the limited resources within which we need to optimise our objective function.

3) All the variables in the objective fx^n and constraint should be linear and non negative.

Laws or rules in LP :-

i) **Law of certainty** :-

In LP model the various parameter like objective fx^n coefficient, constraint, resources are known exactly and their values does not change with time.

ii) **Law of proportionality** :-

iii) **Law of Addition** :- A − 5, B − 3, total → 8

iv) **Law of continuity / Divisibility** :-

In LP model decision variable are continuous i.e. they are permitted to take any non -ve value that satisfy all the constraint.

General Statement of LP :-

$$\text{Max } z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \quad // \text{ obj. function}$$

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m \end{array}\right\} \text{Constraint}$$

Non - negative condition $\rightarrow x_1, x_2, \cdots x_n \ge 0$

where, $a_{ij} \rightarrow$ technological coeff or substitution

$b_i \rightarrow$ resource value

$c_j \rightarrow$ profit coefficient

$x_j \rightarrow$ decision or choice variable.

a_{ij}, b_i, c_j are const and x_j is variable.

Graphical Method :-

Steps -

i) Identify the problem and define decision variable, objective fx^n and constraint.

ii) Draw a graph that include all the constraint and identify the common feasible region.

iii) Find out the point within feasible region that optimises the objective fx^n, this point gives the final solution

P.83, Q 33 →

mc → / product ↓	P	Q	R	Profit/unit
A	10	6	5	60
B	7.5	9	13	70
Max hrs/week	75	54	65	

1) Key decision is to determine no. of units produced of product A and product B in a week. Let these are x_1 and x_2 resp.

2) Feasible alternatives are all the values of $x_1, x_2 \geq 0$

3) objective is to maximise weekly profit when the profit per unit is given so the objective function

$$\text{Max } Z = 60x_1 + 70x_2$$

4) Restriction is of the max. ~~tin~~ m/cing time available for the 3 m/c in a week so the constraint are –

$$P \rightarrow \quad 10x_1 + 7.5x_2 \leq 75$$

$$Q \rightarrow \quad 6x_1 + 9x_2 \leq 54$$

$$R \rightarrow \quad 5x_1 + 13x_2 \leq 65$$

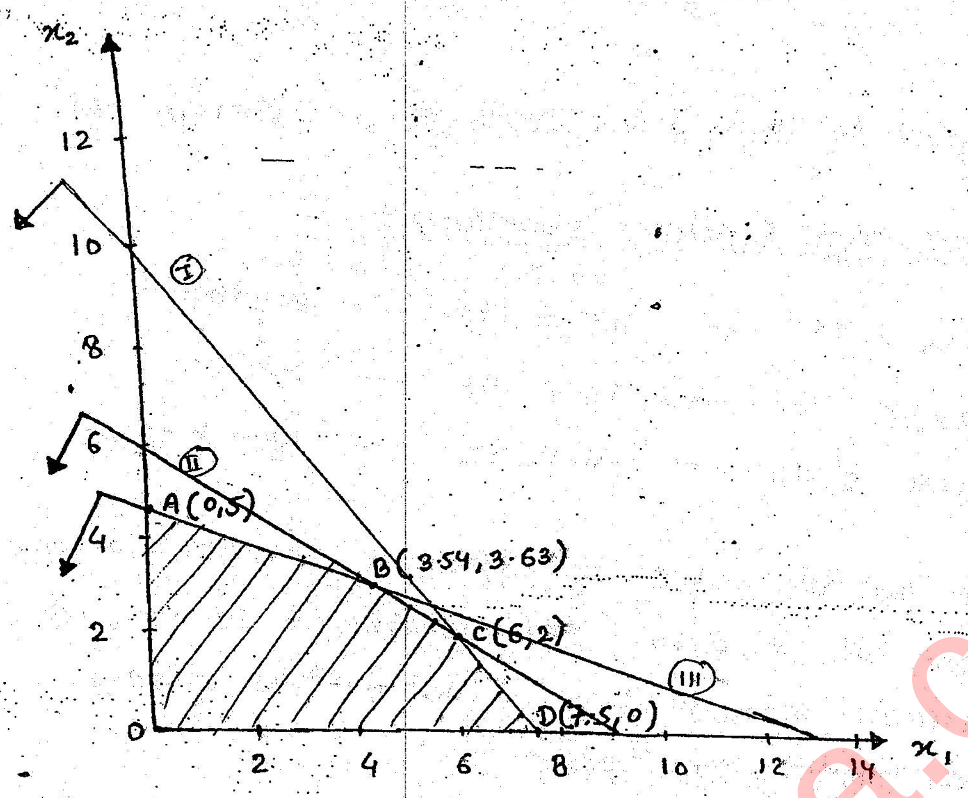
5) All the constraint are plotted on the graph to get feasible region

$$10x_1 + 7.5x_2 = 75$$

$$\frac{x_1}{7.5} + \frac{x_2}{10} = 1$$

$$\frac{x_1}{9} + \frac{x_2}{6} = 1$$

$$\frac{x_1}{13} + \frac{x_2}{5} = 1$$



6) Shaded region OABCD is the region of feasible sol^n and any point within this region can be our sol^n under the given constraint.

7) optimility – now put the values of corner point at the feasible region in the obj. fx^n. The point which optimises the obj. fx^n gives the final sol^n.

$$Z(A) = 60 \times 0 + 70 \times 5 = 350$$

$$Z(B) = 60 \times 3.54 + 70 \times 3.63 = 466.5$$

$$Z(C) = 60 \times 6 + 70 \times 2 = 500$$

$$Z(D) = 60 \times 7.5 + 70 \times 0 = 450$$

$$Z(O) = 60 \times 0 + 70 \times 0 = 0$$

$\text{Max} \rightarrow Z = 500$

at $x_1 = 6$ and $x_2 = 2$

one of the vertex of the feasible region give the final sol^n b,coz obj. fx^n is a straight line with const slope and as it move away from the origin its value increases and optimum value will be at one of the corner extreme point Obj. function will be tangent to that point and give optimum solⁿ.

Binding and Non Binding Constraint :-

at $x_1 = 6$, $x_2 = 2$

$P \rightarrow 10x_1 + 7.5x_2 \leq 75 \longrightarrow 75 = 75 \longrightarrow$ Binding

$Q \rightarrow 6x_1 + 9x_2 \leq 54 \longrightarrow 54 = 54 \longrightarrow$ Binding

$R \rightarrow 5x_1 + 13x_2 \leq 65 \longrightarrow 56 < 65 \longrightarrow$ Non binding.

when we put the values of optimum solution in the constraint and LHS = RHS, the constraint is termed as Binding otherwise non-binding. Final solⁿ is always obtained from binding constraint.

Q- Solve the following LP problem for minimisation -

$$\text{Min } Z = 6x_1 + 4x_2$$

$$\cancel{3x_1 + 3x_2 \geq 40}$$

$$3x_1 + x_2 \geq 40$$

$$2x_1 + 5x_2 \geq 44$$

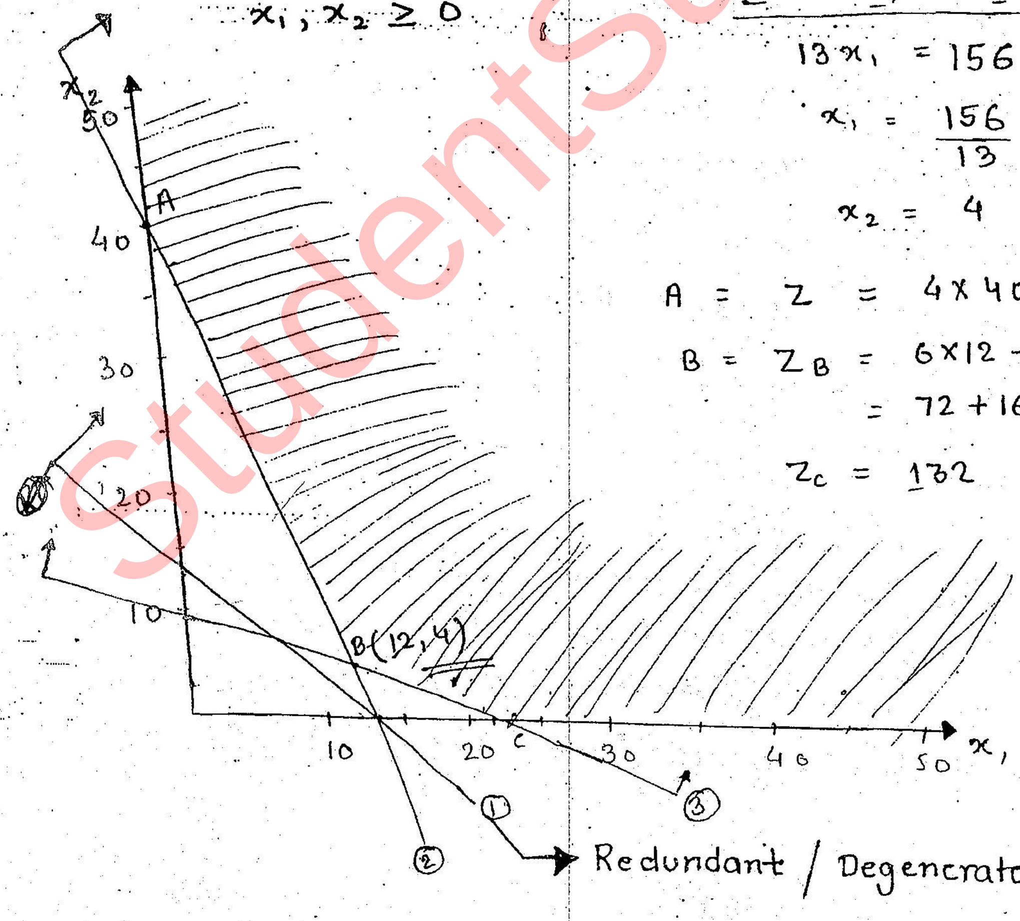
$$x_1, x_2 \geq 0$$

$$3x_1 + x_2 = 40 \times 5$$
$$2x_1 + 5x_2 = 44$$
$$15x_1 + 5x_2 = 200$$

$$13x_1 = 156$$

$$x_1 = \frac{156}{13} = 12$$

$$x_2 = 4$$



$A = Z = 4 \times 40 = 160$

$B = Z_B = 6 \times 12 + 4 \times 4 = 72 + 16 = 88$

$Z_C = 132$

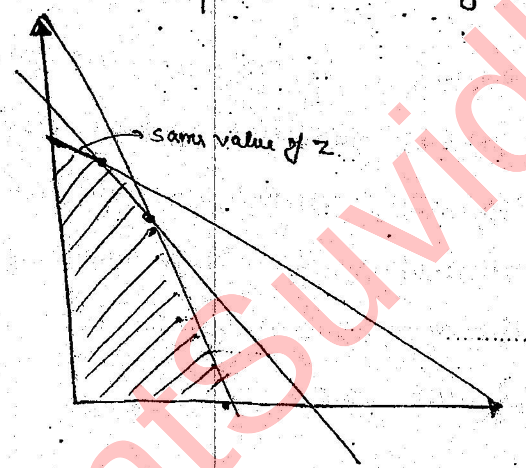
Min $Z = 88$

Redundant :- Constraint which does not become part of the boundary making feasible region is termed as redundant or unnecessary constraint. Inclusion or exclusion of such constraint does not have any effect on optimum final solution of the problem.

Special Cases →

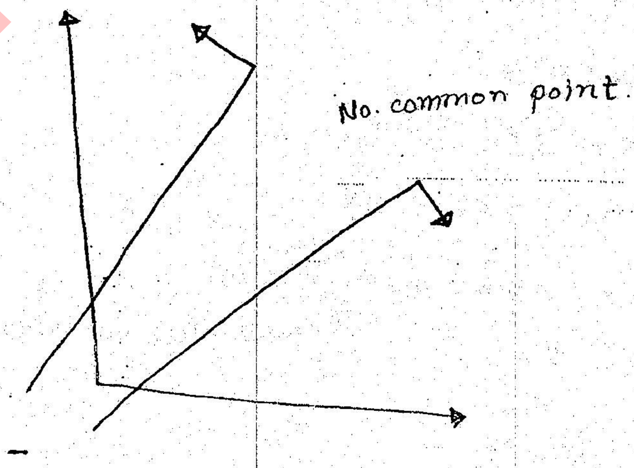
Infinite / Multi optimum solution :-

Infinite no. of solution means we get the same optimum value of objective fx^n for different varying variable. We always get a unique sol^n when the slope of obj. fx^n is different from constraint. Infinite no. of sol^n is obtained when the slope of objective fx^n become equal to one of the binding constraint.

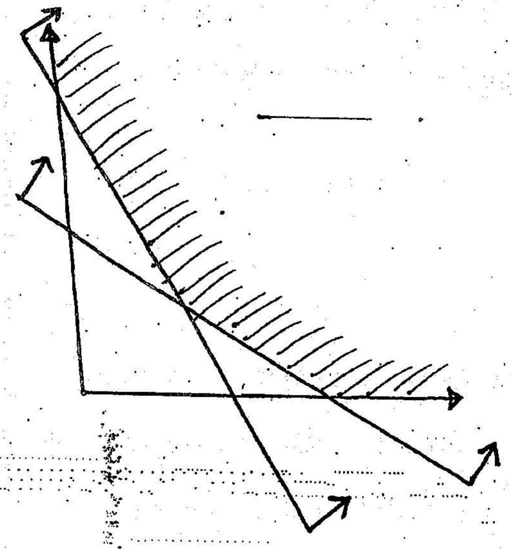


No Solution or Infeasible :-

In some condition constraint may be inconsistent in such a manner that it is not possible to find a feasible sol^n which satisfy all the constraint. There is no solution to such problem.



3) Unbounded Solution :-



In some condition the highest value of obj $f x^n$ goes upto infinite and it simply means that the common feasible region is not bounded by the limit on the constraint. It is termed as unbounded solution.

Simplex Method :-

It is step by step procedure in which we proceed in systematic manner from initial feasible ~~main~~ sol^n with an improve upon the initial solution ~~untill~~ in certain no. of steps we reach the final optimum sol^n. This method also check the corner point of the feasible region but in multidimension depending upon the no. of variable

Standard form of simplex :-

1) All the resource value for the given constraint should be non negative

2) All the inequalities of the given constraint should be converted into equalities.

$$2x_1 - 3x_2 \le -45$$

$$-2x_1 + 3x_2 \ge 45$$

$$-2x_1 + 3x_2 - S_1 = 45$$

S_1 → surplus variable.

or $3x_1 + x_2 \le 40$

$$3x_1 + x_2 + S_1 = 40$$

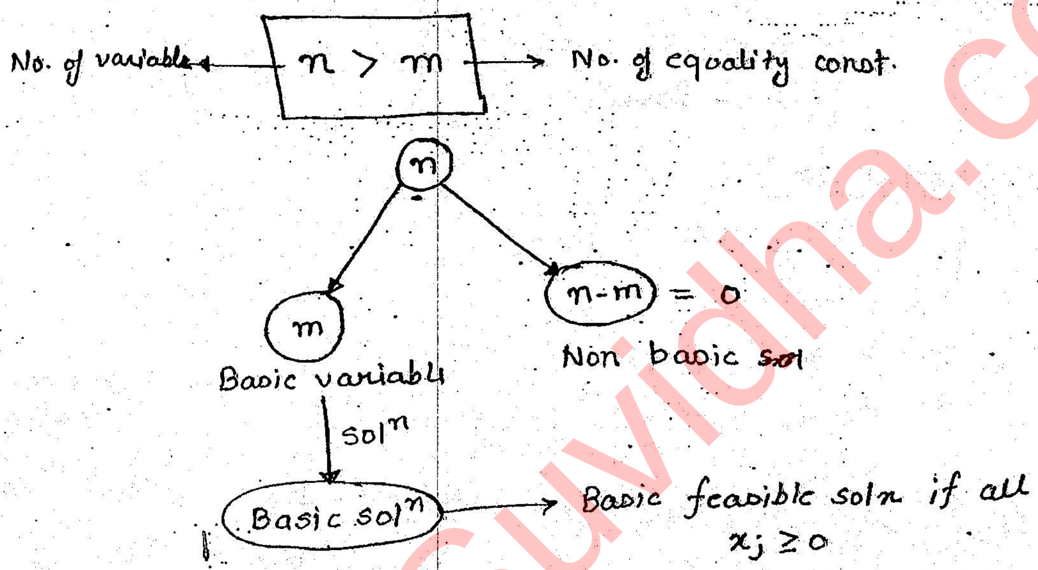
S_1 → slack variable

3) each of the decision variable for the constraint and objective function should be non negative and linear.

$$\boxed{x_j \geq 0}$$

$$\text{Max } z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &\leq b_2 \\ &\vdots \\ a_{mn}x_m &\leq b_m \end{aligned}\right\} \text{constraint.}$$



If there are 'm' equality constraint and 'n' is the no. of variable and $n > m$, then we need to put $(n-m)$ variable $= 0$ known as non basic variable and solve the remaining 'n' basic variable to give basic sol^n. This step reduces the no. of alternate sol^n whose max limit is given by –

$$\boxed{{}^{n}C_m = \frac{n!}{m!\,(n-m)!}}$$

$n = 5$
$m = 3$
$(n-m) = 2 \longrightarrow 0$

P83, Q35 :- Max $Z = 40x_1 + 35x_2$

Max $Z = 40x_1 + 35x_2 + 0S_1 + 0S_2$

R.M $\rightarrow 2x_1 + 3x_2 \leq 60$

Labour $\rightarrow 4x_1 + 3x_2 \leq 96$

$n = 4$ | $2x_1 + 3x_2 + S_1 = 60$

$m = 2$ | $4x_1 + 3x + S_2 = 96$

$n - m = 2$ | $x_1 = 0 ; x_2 = 0$

(Non - basic)

1st feasible sol^n -

$x_1 = 0, x_2 = 0$

$S_1 = 60$

$S_2 = 96$

$Z = 0$

c_i	Basic	x_1	x_2	S_1	S_2	b_i	Replacement Ratio θ_i (b_i/a_{ij})
0	S_1	2	3	1	0	60	30 $(60/2)$
0	S_2	4	3	0	1	96	24 $\leftarrow$ Key Row
	C_j	40	35	0	0		
	$Z_j = \Sigma c_i a_{ij}$	0	0	0	0		
	$\Delta_j = C_j - Z_j$	40	35	0	0		
		$\uparrow$ Key column					

calculate Δj value as a diff. of C_j and Z_j row and it is termed as net evaluation row or Net opportunity cost row. The value of Δj row give the amount of increase or decrease in the objective fx^n that would occur if one unit represented by the column head is brought into the current sol^n. A simplex table indicate that the current sol^n is optimum when all the values in Δj row are -

i) Negative or zero when LP is for maximisation.

ii) Positive or zero when LP is for minimisation.

The current problem is for maximisation so we select the highest +ve value in Δj row and the selected column is called Key column with the variable in the column head as incomming variable. Now divide b_i values by corresponding ~~val~~ elements of key column to get replacement ratio column. In this column we select the min positive value, selected row is called Key row and variable in the row as outgoing variable. The element at the intersection of Key column and key row is termed as key element. Key element is converted to unity by multiplying or dividing key row by a common multiplying factor. All the element in the key column are made zero except key element which will be unity. This is done by adding or subtracting the proper multiples of key row from other row. In the new table outgoing variable is replaced by incoming variable

c_i	basic	x_1	x_2	s_1	s_2	b_i
0	s_1	0	3/2	1	-1/4	12
40	x_1	1	3/4	0	1/4	24

$R_2 \rightarrow R_2 \div 4$

$R_1 \rightarrow R_1 - 2R_2$

2nd feasible sol^n -

$x_1 = 24$, $x_2 = 0$

$S_1 = 12$, $S_2 = 0$

$Z = \text{Rs } 960$

c_i	basic	x_1	x_2	s_1	s_2	b_i	$\theta_i = \frac{b_i}{a_{ij}}$
0	s_1	0	3/2	1	−1/2	12	8 ←
40	x_1	1	3/4	0	1/4	24	32
	c_j	40	35	0	10		
	$z_j = \sum e_i a_{ij}$	40	30	0	10		
	$\Delta_j = c_j - z_j$	0	5 ↑	0	−10		

$R_1 \to R_1 \times \frac{2}{3}$

$R_2 \to R_2 - \frac{3}{4} R_1$

c_i	Basic	x_1	x_2	s_1	s_2	b_i
35	x_2	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	8
40	x_1	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	18

3rd feasible soln –

$x_1 = 18$, $x_2 = 8$

$s_1 = 0$, $s_2 = 0$

$Z = \text{Rs } 1000$

c_i	basic	x_1	x_2	s_1	s_2	b_i	θ_i
35	x_2	0	1	2/3	$-\frac{1}{3}$	8	
40	x_1	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	18	
	c_j	40	35	0	0		
	$z_j = \sum e_i a_{ij}$	40	35	10/3	25/3		
	$\Delta_j = c_j - z_j$	0	0	$-\frac{10}{3}$	$-\frac{25}{3}$		

Big - M Method :-

$\boxed{\geq \text{ or } =}$

$2x_1 - 3x_2 \geq 75$

$2x_1 - 3x_2 - S_1 = 75$

$\hookrightarrow x_1 = 0, x_2 = 0, S_1 = -75$ ✗

$\therefore 2x_1 - 3x_2 - S_1 + A_1 = 75$

Non - basic

Basic	x_1	x_2	S_1	S_2	A_1	b_i
A_1						
S_2						
Δj						✗

max $= -MA_1$ ✓

min $= +M.A_1$ ✓

It is a modified form of simplex method and is always require whever the constrains are $\geq$ or $=$ type, irrespective of whether the problem is for maximisation or for minimisation. In these condition we introduce an artificial variable in the current sol^n to get an initial working matrix. These artificial variable must not appear in the final solution and this is ensured by providing an extremely -ve value to their profit coefficient in the objective fx^n.

Max $= -MA_1$

Min $= +MA_1$

where, m is the no. higher than any infinite number.

Special Cases :-

Infinite or Multioptimum Solution :-

When a non basic variable in an optimum solution has zero value for Δj row then the solution is not unique and it indicates that the problem has infinite no. of solutions.

Unbounded Solution :-

If in a case all the values in the replacement ratio column are either -ve or infinite then the solution terminates and it indicates that the problem has unbounded solution.

θ_i
-ve or ∞

No Solution or Infeasibility :-

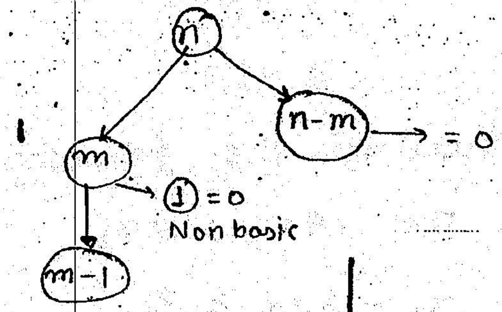
when in the final solution artificial ~~solution~~ variable remains in the basis, then there is no feasible sol^n to the problem.

Basis	x_1	x_2	s_1	s_2	A_1
A_1		$\geq$ or =			
s_2					

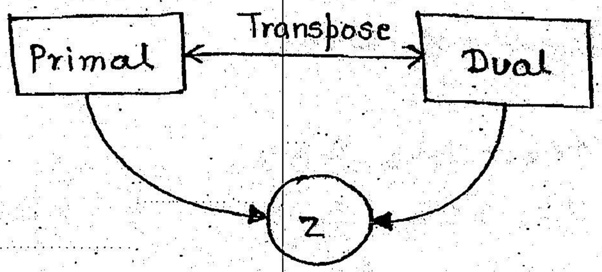
Degenerate Solution :-

When one or more of the basic variable becomes equal to zero during calculation, the solution is called degenerate and the condition is known as degeneracy.
In a degenerate sol^n the no. of basic variable becomes less than equality constraint.

$$\boxed{\begin{matrix}4\\10\\4\end{matrix}}$$



Duality :-



The initial given problem is known as primal and the problem obtained by transposing row and column but having the same optimum value of objective fx^n is called as dual.

1) maxim. $\xrightarrow{\cancel{\text{change to}}}$ $\leq$ type constraint

2) min $\longrightarrow$ $\geq$ type constraint

Primal		Dual
Maximum	$\longleftrightarrow$	Minimum
n	$\longleftrightarrow$	m
m	$\longleftrightarrow$	n
b_i	$\longleftrightarrow$	c_j
c_j	$\longleftrightarrow$	b_i
$\leq$ type constraint	$\longleftrightarrow$	Non - negative Variable
$=$ type constraint	$\longleftrightarrow$	Unrestricted sign variable.

Q → Find the Dual for the following LP problem –

$$\text{Min } Z = 7x_1 - 9x_2 + 12x_3$$

y_1: $2x_1 - x_2 + 5x_3 \ge 8.$

y_2: $3x_1 - 4x_3 \le 7 \Rightarrow -3x_1 + 4x_3 \ge -7$

y_3: $2x_1 + 3x_3 \ge 15$

$\cancel{4x_1 - 3x_2 + 2x_3 = 10}$

y_4: $4x_1 - 3x_2 + 2x_3 \ge 10$

y_5: $4x_1 - 3x_2 + 2x_3 \le 10 \Rightarrow -4x_1 + 3x_2 - 2x_3 \ge -10$

Dual →

$$\text{Max } W = 8y_1 - 7y_2 + 15y_3 + 10y_4 - 10y_5$$

$2y_1 - 3y_2 + 4y_4 - 4y_5 \le 7$

$-y_1 - 3y_4 + 3y_5 \le -9$

$5y_1 + 4y_2 + 3y_3 + 2y_4 - 2y_5 \le 12$

$y_1, y_2, y_3, y_4, y_5 \ge 0$

earlier we were having 4 equath, 4th eqn is converted into 2 other eqn so converting these 2 eqn back to 1 –

∴ Let $y_4 - y_5 = y_6$ (∵ Coeff. is same).

∴ $\text{Max } W = 8y_1 - 7y_2 + 15y_3 + 10y_6$

$2y_1 - 3y_2 + 4y_6 \le 7$

$-y_1 + 2y_3 - 3y_6 \le -9$

$5y_1 + 4y_2 + 3y_3 + 2y_6 \le 12$

$y_1, y_2, y_3 \ge 0$; y_6 → unrestricted in sign