(iii) Give example of field in which the fundamental theorem is false. Explain.

(iv) State Mertens theorem.

(v) What are magnitude and average order of $\sigma(n)$?

(vi) Define Mobius function.

(vii) Prove that $\sum_{i=1}^{n} i = \frac{n\,\phi(n)}{2}$

(viii) State Bertrand Postulate.

M.Sc. 4th Semester Examination, May–2016

MATHEMATICS

Paper–MM–525 (C–2)

Analytical Number Theory–II

Time allowed : 3 hours] *[Maximum marks : 80*

Note : *Attempt one question from each of the* ***Unit–I, II, III, IV. Unit–V is compulsory.***

Unit–I

1. (a) Using Riemann zeta function, prove that there are infinitely many primes.

(b) Prove that if b (s) is a Riemann zeta function, then evaluate b (2k), where $k \geq 1$.

2. (a) Show that $\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} (\xi(s))^2$ for $s > 1$.

(b) Suppose that $F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$, $G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ and $H(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}$ where f, g, h are arithmetic functions such that $h = f * g$, then

$H(s) = F(s)\,G(s)\ \forall\ s$ such that F (s) and G (s) both converge absolutely.

Also, Using above theorem, show that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\xi(s)}$$

Unit–2

3. (a) State and prove Division Algorithm in Q(i).

(b) Prove that if k is a quadratic field, $\exists$ a non-zero square free integer m such that $k = Q(\sqrt{m})$.

4. Prove that the prime of $Q(\sqrt{2})$ can be divided into 3 classes

(i) The prime $\sqrt{2}$ and its associates.

(ii) The rational primes of the form $8n \pm 3$ and their associates.

(iii) The prime factor $a + b\sqrt{2}$ of the rational primes of the form $8n \pm 1$ and their associates.

Unit–3

5. (a) Prove that d (n) and σ (n) are multiplicative functions. Also, find formulae for d (n) and σ (n).

(b) Prove that if f (n) is a multiplicative function and $f \neq 0$. Then $f(1) = 1$.

6. (a) What is a General principle ? By using this principle prove that

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \ldots \left(1 - \frac{1}{p_k}\right)$$

where n > 1 and let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ where $p_1, p_2, \ldots p_k$ are distinct prime.

(b) Find the sum of the cubes of the integers $\leq n$ and relatively coprime to n where n > 1. Also state clearly results used by you.

Unit–IV

7. (a) Prove that if $x > 0$, then

$$0 \leq \frac{\psi(x)}{x} - \frac{\theta(x)}{x} \leq \frac{(\log x)^2}{2\sqrt{x}\log 2}$$

Also define Chebyshev's functions ψ (x) and θ (x).

(b) State and prove Abel's identity. Also give its applications.

8. (a) Prove that there is a constant A such that

$$\sum_{p \leq x} \frac{1}{p} = \log\log x + A + O\left(\frac{1}{\log x}\right)$$

for all $x \geq 2$.

(b) State and prove Selberg's asymptotic formula.

Unit–V

9. (i) Show that $\lambda = 1 - w$ is a prime and 3 is associated with λ^2.

(ii) What do you understand by Dirichlet's Series ?