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M. Sc. (Mathematics) 4th Semester Examination – December, 2014

ANALYTICAL NUMBER THEORY - II

Paper : MM-525(C2)

Time : Three Hours] *[Maximum Marks : 80*

Before answering the questions, candidates should ensure that they have been supplied the correct and complete question paper. No complaint in this regard, will be entertained after examination.

Note : Attempt *five* questions in all, selecting exactly *one* question from Unit - I, II, III and IV. Unit - V (Q. No. 9) is *compulsory*.

UNIT – I

1. (a) Let σ_a be the abscissa of the absolute convergence of the Dirichlet series $A(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$. If $c > \max(0, \sigma_a)$, then prove that $\sum_{n \le x} |a_n| = 0(x^s)$ as $x \to \infty$ and conversely. 8

(b) Suppose s is a real number for which the Dirichlet series $\sum \frac{a_m}{m^s}$ and $\sum \frac{b_n}{n^s}$ are both absolutely convergent. Let $C_r = \sum_{d/r} a_d \, b_{\frac{r}{d}}$. Then, the Dirichlet series $C(s) = \sum \frac{C_r}{r^s}$ is absolutely convergent. Also, find its sum. 8

2. (a) Prove that for $s > 1$, the following : 8

(i) $\zeta(s) = \frac{1}{s-1} + O(1)$

(ii) $\zeta'(s) = -\frac{1}{(s-1)^2} + O(1)$

holds simultaneously.

(b) If $s > 1$, then prove that $\frac{1}{\zeta(s)} = \sum_{m=1}^{\infty} \frac{\mu(m)}{m^s}$. 8

UNIT – II

3. Let p be a prime such that $p \equiv 3(\text{mod } 4)$. Then, prove that the Mersenne number $M = 2^p - 1$ is a prime iff $\left(\frac{\sqrt{5}+1}{2}\right)^{2^{p-1}} + \left(\frac{1-\sqrt{5}}{2}\right)^{2^{p-1}} \equiv 0\left(\text{mod}(2^p - 1)\right)$. 16

4. Prove that the number of real Euclidean fields $Q(\sqrt{m})$ where m is either of the form $4k + 2$ or of the form $4k + 3$ is finite. 16

UNIT – III

5. (a) If $m > 1$, prove that sum of $\phi(m)$ positive integers which are less than m and relatively prime to m is $\frac{m\phi(m)}{2}$. 8

(b) Prove that for arithmetic functions $\phi(n)$ and $\sigma(n)$, there exists a constant A, such that $A < \frac{\sigma(n)\phi(n)}{n^2} < 1 \; \forall \; n > 1$. 8

6. (a) Prove that there exists no any $A > 0$ such that

$d(n) \leq A \,|\, (\log n)^{\Delta} \,|$ for all $\Delta > 0$. 8

(b) Prove that every even perfect number must be of the form $2^n(2^{n+1} - 1)$, where $2^{n+1} - 1$ is a prime number. 8

UNIT – IV

7. (a) State and prove Merten's Theorem. 8

(b) Prove that $\upsilon(n) < 2n \log 2 \; \forall \; n \geq 1$. 8

8. (a) State and prove Abel's Identity. 8

(b) Prove that $\psi(x) = \upsilon(x) + 0\left(x^{1/2} (\log x)^2\right)$. 8

UNIT – V

9. (a) Prove that $\phi(m^2) = m\,\phi(m)$ for every positive integer m.

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(b) Prove that $\sum_{n=1}^{\infty} \mu(\lfloor n) = 1$.

(c) Find the value of $\psi(10)$.

(d) State Prime Number Theorem.

(e) Define Riemann Zeta function.

(f) Prove that $\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}$ for all $s > 2$.

(g) Prove that ± 1 and $\pm i$ are only unities of $Q(i)$.

(h) Prove that any Gaussian integer with norm equal to 7 is a prime in $Q(i)$.
