(vii) Let α be a primitive p-th root of unity and $K = F(\alpha)$, where $F = Q$. Show that $G(K, F)$ is an abelian group.

(viii) Define Galois extension and normal extension.

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M. Sc. 1st Sem. (Mathematics)

Examination – December, 2015

ADVANCED ABSTRACT ALGEBRA-I

Paper : MM-411

Time : Three Hours] *[Maximum Marks : 80*

Before answering the questions, candidates should ensure that they have been supplied the correct and complete question paper. No complaint in this regard, will be entertained after examination.

Note : Attempt *one* question from Units I, II, III and IV. Unit V is *compulsory*.

UNIT – I

1. (a) Define $G^{(n)} (n \geq 0)$ the nth derived subgroup of G. Prove that a group G is solvable if and only if $G^{(n)} = \{e\}$ for the same $n \geq 0$. 8

(b) Let H be a normal subgroup of a group G such that H and G/H both are solvable. Prove that G is solvable. 8

2. (a) Show that all the composition series for a group G, having a composition series are equivalent. 8

(b) State and prove the Schreier's refinement theorem. 8

UNIT – II

3. (a) Let $G = G_0 \supseteq G_1 \supseteq .... \supseteq G_n = \langle e \rangle$ be a normal series for G. Show that this series is a central series if and only if $[G_{i-1}, G] \subseteq G_i$ for $1 \le i \le n$. 6

(b) Show that finite direct product of nilpotent groups is again nilpotent. 6

(c) Show that proper subgroup of a nilpotent group is also a proper subgroup of its normalizer. 4

4. (a) Let P be a Sylow p-subgroup of G and $x \in N(p)$ such that $O(x) = p^i$ for same $i \ge 0$. Show that $x \in p$. 6

(b) Characterize all the groups of order pq, where p and q are primes. Deduce that a group of order 15 is always cyclic. 10

UNIT – III

5. (a) Determine the splitting field K for the polynomial $x^5 - 3x^3 + x^2 - 3$ over Q. Also find $[K : Q]$ and a basis for K/Q. 8

(b) Let $f(x) \in F[x]$ be any polynomial of degree n over F. Prove that there exists an extension E of F which contains all the roots of $f(x)$ and $[E : F] \le \underline{|n}$. 8

6. (a) Let Q be the field of rationals, then find a $u \in Q(\sqrt{2}, \sqrt{3})$ such that $Q(\sqrt{2}, \sqrt{3}) = Q(u)$. 6

(b) Show that $\sin m°$ is an algebraic integer for every integer m. 6

(c) Define a perfect field and show that every algebraic extension of a perfect field is separable extension. 4

UNIT – IV

7. (a) Show that every finite extension of a finite field is normal extension. 8

(b) For every prime p and integer $n \ge 1$, show that there exists a field having p^n elements. 8

8. (a) State and prove Artin's theorem. 8

(b) Find the Galois group of the polynomial $x^3 - 2$ over Q the field of rationals. 8

UNIT – V

9. (i) Show that a composition series has no proper refinement. 16

(ii) Give example of a subnormal series which is not a normal series.

(iii) Show that any group G with $G' = Z(G)$ is nilpotent of class 2, where G' is derived subgroup of G and $Z(G)$ is centre of G.

(iv) Define upper and lower central series.

(v) Give example of an inseparable extension.

(vi) If $a \in k$ is algebraic over F, then show that a and $\sigma(a)$ are conjugates, where σ is an F-automorphism on k.