B.E.(M.D.U.)

First Semester Examination, Dec-2008

Mathematics-1 (Math-101-E)

Note : Attempt ***FIVE*** questions, Selecting at least two from each part.

Part-A

Q. 1. (a) Test the convergence or divergence of the series :

$$\sum_{n=1}^{\infty} \frac{1}{n^p(n+1)^p}.$$

Ans. Here

$$a_n = \frac{1}{n^p(n+1)^p}$$

$$a_{n+1} = \frac{1}{(n+1)^p(n+2)^p}$$

$$\frac{a_n}{a_{n+1}} = \frac{(n+2)^p}{n^p} = \left(1+\frac{2}{n}\right)^p \qquad \text{...(1)}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1$$

D' Alemberts Ratio test fail

From equation (1),

$$\frac{a_n}{a_{n+1}} = \left(1+\frac{2}{n}\right)^p$$

$$= 1+\frac{2p}{n}+\frac{p(p-1)}{2!}\frac{4}{n^2}+\ldots\ldots$$

$$= 1+\frac{2p}{n}+\frac{2p(p-1)}{n^2}+\ldots\ldots$$

$$= 1+\frac{2p}{n}+0\left(\frac{1}{n^{1+\delta}}\right)$$

Where stands for the terms of highest powers of $\frac{1}{n}$.

Thus,

$$\frac{a_n}{a_{n+1}} = 1+\frac{2p}{n}+0\left(\frac{1}{n}1+\delta\right)$$

$\therefore$ By Gauss test, the series is convergent if $2p > 1$ i.e., $p > \frac{1}{2}$ and divergent if $2p \le 1$ i.e. $p \le \frac{1}{2}$.

Q. 1. (b) Discuss the convergence of the series :

$$\frac{1^2}{2^2}+\frac{1^2.3^2}{2^2.4^2}+\frac{1^2.3^2.5^2}{2^2.4^2.6^2}+\ldots\ldots\ldots$$

Ans. Here

$$u_n=\frac{1^2.3^2.5^2\ldots\ldots(2n-1)^2}{2^2.4^2.6^2\ldots\ldots(2n)^2}$$

$$u_{n+1}=\frac{1^2.3^2.5^2\ldots\ldots(2n-1)^2(2n+1)^2}{2^2.4^2.6^2\ldots\ldots(2n)^2(2n+2)^2}$$

$$\therefore \quad \frac{u_{n+1}}{u_n}=\frac{(2n+1)^2}{(2n+2)^2}=\frac{un^2\left(1+\frac{1}{2n}\right)^2}{un^2\left(1+\frac{1}{n}\right)^2}=\frac{\left(1+\frac{1}{2n}\right)^2}{\left(1+\frac{1}{n}\right)^2}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=1$$

D'Alembert's Ratio Test fails. Let us try Raabe's Test,

$$\frac{u_n}{u_{n+1}}=\frac{(2n+2)^2}{(2n+1)^2}$$

$$\frac{u_n}{u_{n+1}}-1=\frac{(2n+2)^2-(2n+1)^2}{(2n+1)^2}=\frac{u_{n+3}}{(2n+1)^2}$$

$$n\left(\frac{u_n}{u_{n+1}}-1\right)=\frac{n(un+3)}{(2n+1)^2}=\frac{un^2+3n}{(2n+1)^2}=\frac{un^2\left(1+\frac{3}{un}\right)}{un^2\left(1+\frac{1}{2n}\right)^2}$$

$$=\frac{\left(1+\frac{3}{un}\right)}{\left(1+\frac{1}{2n}\right)^2}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=1$$

The test fails. Let us try Gauss test,

$$\frac{u_n}{u_{n+1}}=\frac{(2n+2)^2}{(2n+1)^2}=\frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{1}{2n}\right)^2}=\left(1+\frac{1}{n}\right)^2\left(1+\frac{1}{2n}\right)^2$$

$$= \left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left(1-\frac{2}{2n}+\frac{3}{4n^2}-\frac{4}{8n^3}+\ldots\ldots\right)$$

$$= 1+\frac{1}{n}+\frac{1}{n^2}\left(1-2+\frac{3}{4}\right)+\ldots\ldots$$

Comparing with Gauss Test Formula,

$$\frac{u_n}{u_{n+1}} = 1+\frac{l}{n}+\frac{c_n}{n^2}$$

Here $l = 1$

$\therefore$ The series is divergent.

Q. 1. (c) Discuss the convergence of the series :

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p},\ p > 0.$$

Ans.
$$a_n = \frac{1}{n(\log n)^p},\ p > 0$$

Let
$$f(x) = \frac{1}{x(\log x)^p},\ p > 0$$

$\therefore$ $f(n) = a_n$ for all $n \geq 2$

Consider $x_1 < x_2$ where $x_1, x_2 \geq 2$

$\therefore$
$$x_1(\log x_1)^p < x_2(\log x_2)$$

Or
$$\frac{1}{x_1[\log(x_1)]^p} > \frac{1}{x_2(\log x_2)^p}$$

Or
$$f(x_1) > f(x_2)$$

$\Rightarrow$ f(x) is a decreasing function of x. Also f(x) is non-negative for all $x \geq 2$.

$\therefore$ Cauchy's integral test is applicable.

Case I : When $p > 1$

$$I_n = \int_2^n f(x)dx = \int_2^n (\log x)^{-p}\frac{1}{x}dx = \left[\frac{(\log x)^{1-p}}{1-p}\right]_2^n$$

$$= \frac{1}{1-p}\left[\frac{1}{(\log x)^{p-1}}\right]_2^n = \frac{1}{1-p}\left[\frac{1}{(\log n)^{p-1}} - \frac{1}{(\log 2)^{p-1}}\right]$$

$$\lim_{n\to\infty} I_n = \lim_{n\to\infty} \frac{1}{1-p}\left[\frac{1}{(\log n)^{p-1}} - \frac{1}{(\log 2)^{p-1}}\right]$$

$$= \frac{1}{1-p}\left[0 - \frac{1}{(\log 2)^{p-1}}\right] = \frac{1}{(p-1)(\log 2)^{p-1}}$$

= Which is a finite quantity.

$\therefore$ Sequence $\langle I_n \rangle$ converges.

$\therefore$ By Cauchy Integral test, $\sum_{n=1}^{\infty} a_n$ is convergent.

Case II : If $p < 1$.

$$I_n = \int_2^n f(x)\,dx = \int_2^n (\log x)^{-p} \frac{1}{x}\,dx = \left[\frac{(\log x)^{1-p}}{1-p}\right]_2^n$$

$$= \frac{1}{1-p}\left[(\log n)^{1-p} - (\log 2)^{1-p}\right]$$

$$\lim_{n\to\infty} I_n = \lim_{n\to\infty} \frac{1}{1-p}\left[(\log n)^{1-p} - (\log 2)^{1-p}\right]$$

$$= \frac{1}{1-p}\left[\infty - (\log 2)^{1-p}\right] = \infty$$

$\therefore$ Sequence $\langle I_n \rangle$ is divergent and hence $\sum_{n=1}^{\infty} a_n$ is divergent.

Case II : If $p = 1$.

$$I_n = \int_2^n \frac{dx}{x\log x} = \int_2^n \frac{\frac{1}{x}}{2\log x}\,dx$$

$$= [\log.\log x]_2^n$$

$$= \log(\log n) - \log(\log 2)$$

$$\lim_{n\to\infty} I_n = \infty - \log(\log 2) = \infty$$

$\therefore$ Sequence $\langle I_n \rangle$ diverges to $+\infty$.

$\therefore$ By Cauchy's integral test, the given series is diverges.

Hence, the given series converges if $p > 1$ and diverges if $p \le 1$.

Q. 2. (a) Find the radius of curvature at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ of Folium $x^3 + y^3 = 3axy$.

Ans. Writing the equation in the form,

$$x^3 + y^3 - 3axy = 0$$

Here,
$$f = x^3 + y^3 - 3axy$$

$$p = \frac{\delta f}{\delta x} = 3(x^2 - ay)$$

$$q = \frac{\delta f}{\delta y} = 3(y^2 - ax)$$

$$r = \frac{\delta^2 f}{\delta x^2} = 6x$$

$$s = \frac{\delta^2 f}{\delta x.\delta y} = -3a$$

$$t = \frac{\delta^2 f}{\delta y^2} = 6y$$

At the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ we have,

$$p = \frac{9a^2}{4},\ q = \frac{9a^2}{4},\ r = 9a,\ s = -3a,\ t = 9a$$

Now,
$$\rho = \frac{(p^2 + q^2)^{3/2}}{-q^2 r + 2pqs - p^2 t}$$

Putting the value of p, q, r, s and t, we get,
Radius of curvature

$$(\rho) = \frac{\left(\frac{81}{16}a^4 + \frac{81}{16}a^4\right)^{3/2}}{-\left[\frac{81}{16} \times 9 + 2 \times \frac{81}{16} \times 3 + \frac{81}{16} \times 9\right]a^5}$$

$$= -\frac{\left(\frac{81}{8}a^3\right)^{3/2}}{\frac{81}{6} \times 9a^5} = -\frac{\frac{81}{8}a^4 \times \frac{9}{2\sqrt{2}}a^2}{\frac{81}{6} \times 9a^5}$$

$$= \frac{-3a}{8\sqrt{2}} \text{ the required result.}$$

Q. 2. (b) Find the asymptotes of the curve :

$$y^3 - 2xy^2 - x^2y + 2x^3 + 3y^3$$
$$-7xy + 2x^2 + 2y + 2x + 1 = 0.$$

Ans. It is a third degree equation in x and y. Both x^3 and y^3 are present, therefore, there is no asymptote parallel to either of the axes.

Putting x = 1, y = m in 3rd degree terms, two degree terms and first degree terms, we get,

$$\phi_m(m) = 2 - m - 2m^2 + m^3$$

$$\therefore \qquad \phi_3^1(m) = -1 - 4m + 3m^2$$

$$\phi_2(m) = 2 - 7m + 3m^2$$

$$\phi_1(m) = 2 + 2m$$

For the asymptotes, $\phi_3(m) = 0$

$$m^2(m-2) - (m-2) = 0$$

Or $(m-2)(m^2-1) = 0$

Which gives m = 2, 1, −1.

$\therefore$ C is given by
$$C = \frac{-\phi_2(m)}{\phi_3^1(m)} = -\frac{3m^2 - 7m + 2}{3m^2 - 4m - 1}$$

$\therefore$ When m = 2,
$$C = -\frac{12 - 14 + 2}{12 - 8 - 1} = 0$$

m = 1,
$$C = -\frac{3 - 7 + 2}{3 - 4 - 1} = -\frac{2}{2} = -1$$

m = −1,
$$C = -\frac{3 + 7 + 2}{3 + 4 - 1} = -2$$

$\therefore$ The asymptotes are ,

$$y = 2x, \; y = x - 1 \text{ and } y = -x - 2.$$

Q. 3. (a) If $u = f(r)$ and $x = r\cos\theta$, $y = r\sin\theta$, then prove that :

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r).$$

Ans. Here $u = f(r)$

and $x = r\cos\theta,\ y = r\sin\theta$

Squaring and adding, $x^2 + y^2 = r^2$

$$r = \sqrt{x^2 + y^2} = \left(x^2 + y^2\right)^{1/2}$$

$$\frac{\delta u}{\delta x} = f'(r)\frac{\delta r}{\delta x} = f'(r)\frac{1}{2}\left(x^2 + y^2\right)^{-1/2}.2x$$

$$= f'(r)\frac{x}{\sqrt{x^2 + y^2}}$$

$$= f'(r).\frac{x}{r} \qquad \left[\because \sqrt{x^2 + y^2} = r\right]$$

$$\frac{\delta^2 u}{\delta x^2} = f''(r)\frac{\delta r}{\delta x}\left(\frac{x}{r}\right) + f'(r)\frac{r.1 - x\frac{\delta r}{\delta x}}{r^2}$$

$$= f''(r).\frac{x}{r}.\frac{x}{r} + f'(r)\frac{r - \frac{x.x}{r}}{r^2}$$

$$= \frac{x^2}{r^2} f''(r) + f'(r).\frac{r^2 - x^2}{r^3} \qquad ...(i)$$

Now $$\frac{\delta r}{\delta y} = \frac{1}{2}\left(x^2 + y^2\right)^{-1/2}(2y) = \frac{y}{r}$$

and $$\frac{\delta u}{\delta y} = f'(r)\frac{\delta r}{\delta y}$$

$$= f'(r).\frac{y}{r}$$

$$\frac{\delta^2 u}{\delta y^2} = f''(y)\frac{\delta r}{\delta y}.\left(\frac{y}{r}\right) + f'(r)\frac{r.1 - y.\frac{\delta r}{\delta y}}{r^2}$$

$$= f''(y).\frac{y}{r}.\frac{y}{r} + f'(r)\frac{r - y.\frac{y}{r}}{r^2}$$

$$= f''(y)\frac{y^2}{r^2} + f'(r)\frac{r^2 - y^2}{r^3} \qquad ...(ii)$$

Adding equations (i) and (ii), we get,

$$\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}=f''(r)\left[\frac{x^2}{r^2}+\frac{y^2}{r^2}\right]+\frac{f'(r)}{r^3}\left[r^2-x^2+r^2-y^2\right]$$

$$=f''(r)\left[\frac{x^2+y^2}{r^2}\right]+\frac{f'(r)}{r^3}\left[2r^2-\left(x^2+y^2\right)\right]$$

$$=f''(r)+\frac{f'(r)}{r^3}.\left[2r^2-r^2\right] \qquad \left[\because x^2+y^2=r^2\right]$$

$$=f''(r)+\frac{f'(r)}{r^3}.r^2$$

$$=f''(r)+\frac{1}{r}f'(r) \text{ the required result.}$$

Q. 3. (b) Given $z=x^n f_1(y|x|)+y^n f_2(x|y)$, prove that :

$$x^2.\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial y^2}+x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=n^2 z$$

Ans. Here z is a homogeneous function of degree n in x and y, then,

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz \qquad \text{...(i)}$$

[By Euler's Theorem]

Differentiating both sides partially w.r.t.'x', we have

$$x\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2}{\partial x}+y\frac{\partial^2 z}{\partial x\partial y}=n\frac{\partial z}{\partial x}$$

$$x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial x\partial y}=(n-1)\frac{\partial z}{\partial x} \qquad \text{...(ii)}$$

Differentiating Again (i) partially w.r.t. 'y', we have

$$x\frac{\partial^2 z}{\partial y\partial x}+y\frac{\partial^2 z}{\partial y^2}+\frac{\partial z}{\partial y}.1=n\frac{\partial z}{\partial y}$$

$$x\frac{\partial^2 z}{\partial y\partial x}+y\frac{\partial^2 z}{\partial y^2}=(n-1)\frac{\partial z}{\partial y} \qquad \text{...(iii)}$$

Multiplying equation (ii) by x and equation (iii) by y and then adding,

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=(n-1)\left[x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right]$$

$$=(n-1).nz$$

$$= n^2 z - nz$$

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + x \frac{\partial^2}{\partial x} + y \frac{\partial z}{\partial y} = n^2 z.$$

Q. 4. (a) Expand $f(x, y) = \tan^{-1}(y|x)$ in powers of $(x-1)$ and $(y-1)$ up third degree terms. Hence compute f (1.1, 0.9) approximately.

Ans. Here, a = 1, b = 1 and $f(1,1) = \tan^{-1}(1) = \frac{\pi}{4}$

$$f_x = \frac{-y}{x^2 + y^2}, \quad f_x(1,1) = -\frac{1}{2}$$

$$f_y = \frac{x}{x^2 + y^2}, \quad f_y(1,1) = \frac{1}{2}$$

$$f_{xx} = \frac{2xy}{(x^2 + y^2)^2}, \quad f_{xx}(1,1) = \frac{1}{2}$$

$$f_{xy} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \quad f_{xy}(1,1) = 0$$

$$f_{yy} = \frac{-2xy}{(x^2 + y^2)^2}, \quad f_{yy}(1,1) = -\frac{1}{2}$$

$$f_{xxx} = \frac{2y^3 - 6x^2 y}{(x^2 + y^2)^3}, \quad f_{xxx}(1,1) = -\frac{1}{2}$$

$$f_{xxy} = \frac{2x^3 - 6xy^2}{(x^2 + y^2)^3}, \quad f_{xxy}(1,1) = \frac{-1}{2}$$

$$f_{xyy} = \frac{6x^2 y - 2y^3}{(x^2 + y^2)^3}, \quad f_{xyy}(1,1) = \frac{1}{2}$$

$$f_{yyy} = \frac{6xy^2 - 2x^3}{(x^2 + y^2)^3}, \quad f_{yyy}(1,1) = \frac{1}{2}$$

Taylor's expansion of f (x, y) in powers of $(x-1)$ and $(y-1)$ is given by

$$f(x;y) = f(1,1) + \frac{1}{1!}\{(x-1)f_x(1,1) + (y-1)f_y(1,1)\}$$

$$+\frac{1}{2!}\left\{(x-1)^2 f_{xx}(1,1)+2(x-1)(y-1)f_{xy},(1,1)+(y-1)^2 f_{yy}(1,1)\right\}$$

$$+\frac{1}{3!}\left\{(x-1)^3 f_{xxx}(1,1)+3(x-1)^2(y-1)f_{xxy}(1,1)+3(x-1)(y-1)^2 f_{xyy}(1,1)+(y-1)^3 f_{yyy}(1,1)\right\}+..$$

$$\therefore \quad \tan^{-1}\frac{y}{x}=\frac{\pi}{4}+\left\{(x-1)\left(-\frac{1}{2}\right)+(y-1)\left(\frac{1}{2}\right)\right\}+\frac{1}{2!}$$

$$\left\{(x-1)^2.\frac{1}{2}+2(x-1)(y-1)(0)+(y-1)^2\left(-\frac{1}{2}\right)\right\}+\frac{1}{3!}$$

$$\left\{(x-1)^3\left(-\frac{1}{2}\right)+3(x-1)^2(y-1)\left(\frac{-1}{2}\right)+3(x-1)(y-1)^2\frac{1}{2}+(y-1)^3\frac{1}{2}\right\}+.....$$

$$=\frac{\pi}{4}-\frac{1}{2}\{(x-1)-(y-1)\}+\frac{1}{4}\left\{(x-1)^2-(y-1)^2\right\}$$

$$-\frac{1}{12}\left\{(x-1)^3+3(x-1)^2(y-1)-3(x-1)(y-1)^2-(y-1)^3\right\}+......$$

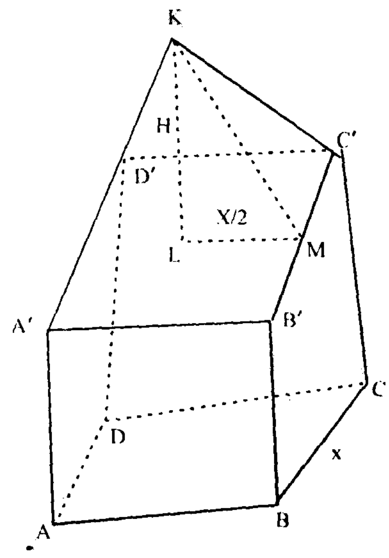
Putting $x = 1.1$ and $y = 0.9$, we get

$$f(1.1,0.9)=\frac{\pi}{4}-\frac{1}{2}(0.2)+\frac{1}{4}(0)-\frac{1}{12}\left\{(0.1)^3-3(0.1)^3-3(0.1)^3-(-0.1)^3\right\}$$

$$=0.7854-0.0967$$

$$=0.6887\cdot$$

Q. 4. (b) A tent on a square base of side x, has its side vertical to height y and the top is a regular pyramid of height h. Find x and y in terms of h, the canvas required for its construction is to be minimum for the tent to have a given capacity.



Ans. Let V be the volume enclosed by the tent and S be its surface area.

Then V = Cuboid (ABCD, A'B'C'D') + Pyramid(K, A'B'C'D')

$$=x^2y+\frac{1}{3}x^2h$$

$$=x^2\left(y+\frac{h}{3}\right)$$

$$S=4(ABGF)+4\Delta KGH$$

$$=4xy+4.\frac{1}{2}(x.KM)$$

$$=4xy+x\sqrt{x^2+4h^2}$$

$$\left[\because KM = \sqrt{KL^2 + LM^2}\right.$$

$$= \sqrt{h^2 + \left(\frac{x}{2}\right)^2}$$

For constant V, we have

$$\partial V = 2x\left(y + \frac{h}{3}\right)\partial x + x^2(\partial y) + \frac{x^3}{3}\partial h = 0$$

For minimum S, we have

$$\partial S = \left[4y + \sqrt{x^2 + 4h^2} + x.\frac{1}{2}\left(x^2 + 4h^2\right)^{-1/2}.2x\right]\partial x + 4x\partial y + x.\frac{1}{2}\left(x^2 + 4h^2\right)^{-1/2} 8h\partial h = 0$$

By Lagrange's Method,

$$\left[4y + \sqrt{x^2 + 4h^2} + x^2\left(x^2 + 4h^2\right)^{-1/2}\right] + \lambda 2x\left(y + \frac{h}{3}\right) = 0 \qquad \text{...(i)}$$

$$4x + \lambda x^2 = 0 \qquad \text{...(ii)}$$

$$4hx\left(x^2 + 4h^2\right)^{-1/2} + \lambda.\frac{x^2}{3} = 0 \qquad \text{...(iii)}$$

(ii) Gives $\lambda = -\frac{4}{x}$. Then (iii) becomes,

$$4hx\left(x^2 + 4h^2\right)^{-1/2} - \frac{4x}{3} = 0$$

Or $$x = \sqrt{5}h$$

Now, putting, $x = \sqrt{5}h$, $\lambda = \frac{-4}{x}$ in equation (ii), we get

$$4y + 3h + \frac{5}{3}h - \frac{4}{x}.2x\left(y + \frac{h}{3}\right) = 0$$

Or $$4y + \frac{14}{3}h - \partial y - \frac{8h}{3} = 0$$

$$y = \frac{h}{2}.$$

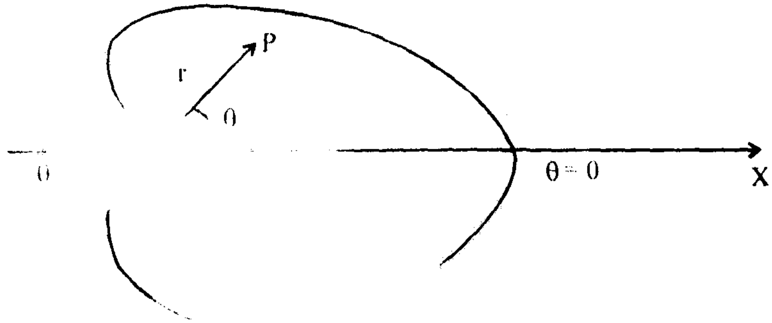
Part-B

Q. 5. (a) Find; by double integration, the volume generated by revolving the cardioid $r = c(1 + \cos\theta)$ about the initial line.

Ans. Equation of the cardioid is $r = a(1 + \cos\theta)$. Let the elementary area be $r\,d\theta\,dr$ or dx dy volume rated by this area about initial line

$$= 2\pi y\, dx\, dy$$

$$= 2\pi(r\, d\theta\, dr)(r \sin\theta)$$



Total volume =

$$\int_0^{\pi}\int_0^{a(1+\cos\theta)} 2\pi r^2 \sin\theta\, dr\, d\theta$$

$$= 2\pi \int_0^{\pi}\left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)}$$

$$= \frac{2\pi}{3}\int_0^{\pi} a^3(1+\cos\theta)^3 .\sin$$

$$= -\frac{2\pi a^3}{3}\int_0^{\pi} (1+\cos\theta)^3(-\sin\theta)$$

$$= \frac{-2\pi a^3}{3}\left|\frac{(1+\cos\theta)^4}{4}\right|_0^{\pi}$$

$$= \frac{-2\pi a^3}{3\times 4}[0-16] = \frac{8\pi a^3}{3}$$

Q. 5. (b) Find the centroid of the region R bounded by the parabolic cylinder $z = 4 - x^2$ and the planes $x = 0, y = 0, y = 6, z = 0$.

Ans.

Q. 6. (a) $\iint \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\, dx\, dy$ over the positive quadrant of the circle $x^2 + y^2 = 1$.

Ans. Let R be the region of integration

$\therefore$ $$R = \{(x, y) : x^2 + y^2 \le 1, x \ge 0, y \ge 0\}$$

$$I = \iint_{x^2+y^2\le 1} \sqrt{\frac{1-(x^2+y^2)}{1+x^2+y^2}}\, dx\, dy \quad ...(i)$$

Let $x = r\cos\theta$

$y = r\sin\theta$

$\therefore$ $x^2 + y^2 = r^2$

But $x^2 + y^2 \le 1$, therefore $r^2 \le 1$ or $0 \le r \le 1$

$$|J| = \begin{vmatrix} \frac{\delta x}{\delta r} & \frac{\delta x}{\delta \theta} \\ \frac{\delta y}{\delta r} & \frac{\delta y}{\delta \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

From equation (1), we have,

$$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} \sqrt{\frac{1-r^2}{1+r^2}}\,|J|\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} \sqrt{\frac{1-r^2}{1+r^2}}\, r\, dr\, d\theta$$

Put $r^2 = z$ so that $2rdr = dz$

When $r = 0$, $z = 0$ and when $r = 1$, $z = 1$

$$= \int_0^{\pi/2}\int_0^1 \sqrt{\frac{1-z}{1+z}}\,\frac{dz}{2}\, d\theta.$$

Q. 6. (b) Find the value of $\Gamma\frac{1}{2}$.

Ans. First, we shall prove that $B(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$

Putting $m = n = \frac{1}{2}$, we get

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)}$$

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \left[\Gamma\left(\frac{1}{2}\right)\right]^2$$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = B\left(\frac{1}{2},\frac{1}{2}\right)$$

$$= \int_0^1 x^{1/2-1}(1-x)^{1/2-1}dx$$

$$= \int_0^1 \frac{1}{\sqrt{x}\sqrt{1-x}}dx$$

Put $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\, d\theta$

When $x = 0,\ \theta = 0$ and when $x = 1,\ \theta = \frac{\pi}{2}$

$\therefore$ From equation (1), we have,

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \int_0^{\pi/2} \frac{1}{\sqrt{\sin^2\theta}\sqrt{1-\sin^2\theta}}(2\sin\theta\cos\theta)d\theta$$

$$= \int_0^{\pi/2} \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta}d\theta = 2\int_0^{\pi/2} d\theta$$

$$= 2[\theta]_0^{\pi/2} = 2\frac{\pi}{2} = \pi$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Q. 7. (a) Find the divergence and curl of the vector :

$\vec{v} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$ at the point (2, –1, 1).

Ans. $\text{div} f = \nabla . f$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left[xyz\hat{i} + 3x^2y\hat{j} + (xz^2 - y^2z)\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z)$$

$$= yz + 3x^2 + 2xz - y^2$$

At point (2, –1, 1)

$$= -1 \times 1 + 3 \times (2)^2 + 2 \times 2 \times 1 - (-1)^2$$

$$= -1 + 12 + 4 - 1$$

$$= 14$$

Curl $f = \nabla \times f$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3xy & xz^2 - y^2 z \end{vmatrix}$$

$$= i(-2yz) - j\left[z^2 - xy\right] + k(6xy - xz)$$

At (2, −1, 1)

$$= (-2 \times -1 \times 1)i - (1+2)j + (-12-2)k$$

$$= 2i - 3j - 14k.$$

Q. 7. (b) If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, then prove that :

$$\nabla^2\left[\nabla\left(\frac{1}{r^2}\right)\right] = 2r^{-4}.$$

Ans.

Q. 8. (a) Evaluate $\iint \vec{A}.\vec{n}\,ds$, where $\vec{A} = z\vec{i} + x\vec{j} - 3y^2z\vec{k}$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

Ans. A vector normal to the surface S is given by

$$\nabla\left(x^2 + y^2\right) = 2x\vec{i} + 2y\vec{j}$$

n = unit vector normal to the surface S at any point of S.

$$= \frac{2x\vec{i} + 2y\vec{j}}{\sqrt{4x^2 + 4y^2}} = \frac{2x\vec{i} + 2y\vec{j}}{2\sqrt{16}} \qquad \left[\because x^2 + y^2 = 16 \text{ on } S\right]$$

$$= \frac{x\vec{i} + y\vec{j}}{4}$$

Now, $\iint \vec{A}.\vec{n}\,ds = \iint_R \vec{A}.\vec{n}\frac{dx\,dz}{|n.\hat{j}|}$, where R is the region of projection of S on xz-plane.

Note that we have not taken projection of S on xy-plane because the surface S is perpendicular to the xy-plane.

$$= \iint_R \left(\frac{xz}{4} + \frac{xy}{4}\right)\frac{dx\,dz}{1/4y}$$

$$= \iint_R \left(\frac{xz + xy}{y}\right)dx\,dz$$

$$= \iint_R \left(\frac{xz}{y} + x\right)dx.dz$$

$$= \int_{z=0}^{5}\int_{x=0}^{4}\left(\frac{xz}{\sqrt{16-x^2}}+x\right)dx$$

$$= \int_0^5\int_0^4\left(\frac{-\frac{1}{2}z(-2x)}{\sqrt{16-x^2}}+x\right)dx$$

$$= \int_0^5\left[\left(-\frac{1}{2}z\right)\frac{\sqrt{16-x^2}}{1/2}+\frac{x^2}{2}\right]_0^4 dz$$

$$= \int_0^5(4z+8)\,dz = \left[\frac{4z^2}{2}+8z\right]_0^5 = 90.$$

Q. 8. (b) Verify Stoke's theorem for the vector field $\vec{F} = (2x-y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$ over he upper half surface of $x^2+y^2+z^2=1$ bounded by its projection on the xy-plane.

Ans. Clearly, the boundary of the upper half of the sphere S is the circle $x^2+y^2=1$ in the xy-plane. the parametric equation of boundary can be taken as $x = \cos t$, $y = \sin t$, $z = 0$, $0 \le t \le 2\pi$

$$\oint f.dr = \oint_C (f_1dx + f_2dy + f_3dz)$$

$$\oint_C \left\{(2x-y)dx - yz^2dy - y^2zdz\right\}$$

$$= \int_0^{2\pi}(2\cos t - \sin t)(-\sin t)\,dt$$

$$= \int_0^{2\pi}\left(-2\cos t\sin t + \sin^2 t\right)dt$$

$$= \left[\cos^2 t\right]_0^{2\pi} + 4\int_0^{\pi/2}\sin^2 t\,dt$$

$$= 0 + 4.\frac{1}{2}.\frac{\pi}{2} = \pi \qquad \text{...(i)}$$

Also curl

$$f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= i(-2yz+2yz) - j(0-0) + k(0+1) = k$$

$$\therefore \quad \iint_S \text{curl}\, f.n\, ds = \iint_S k.n\, ds$$

$$= \iint_R k.n \frac{dx\, dy}{n.k}$$

Where R is the projection of S on xy-plane.

$$= \iint_R dx\, dy$$

$$= \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\, dx$$

$$= 4\int_0^1 \int_0^{\sqrt{1-x^2}} dy\, dx$$

$$= 4\int_0^1 \sqrt{1-x^2}\, dx$$

$$= 4\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right]_0^1$$

$$= 4.\frac{1}{2}.\frac{\pi}{2} = \pi \qquad \text{...(2)}$$

Equality of (1) and (2) verifies Stoke's theorem.