* Introduction

Signals

→ Deterministic — ex- $y(t) = t$

→ Random — Noise etc

* Probability

Experiment → sample space → Event

→ Probability is the study of random variables.

↳ lies between 0 to 1.

$$P(A) = \frac{\text{favourable outcomes}}{\text{Total no. of outcomes}}$$

* Random variable

→ It is basically a rule by which a real number is assigned to each possible outcome of an experiment.

→ If 's' represents the outcome of the experiment, then the random variable is represented by $X(s)$.

→ $X(s)$ is a function that maps the sample points into real numbers $x_1, x_2, x_3 \ldots$ so, we have a random variable X that takes on values $x_1, x_2 \ldots$

* Types of Random variable

→ Discrete

→ Continuous.

Properties of Probability

① $P(A) = 1$

The probability of a certain event is unity

② The probability of any ~~certain~~ event is always less than or equal to 1 and non-negative.

$$0 \leq P(A) \leq 1$$

③ If A and B are two mutually exclusive event then,

$$P(A+B) = P(A) + P(B)$$

④ If A is any event, then the probability of not happening of A is

$$P(\bar{A}) = 1 - P(A)$$

⑤ If A and B are any two events (not mutually exclusive)

$$P(A+B) = P(A) + P(B) - P(AB)$$

Conditional ~~Proper~~ Probability:

Let us consider an experiment which involves two events A and B. Now the probability of event B, given that event A has occured is represented by $P(B/A)$. Similarly $P(A/B)$ represents probability of event A given that event B has already occurred. These are called CONDITIONAL PROBABILITIES.

$$P(B/A) = \frac{P(AB)}{P(A)}$$

* Discrete Random Variable

→ If the sample space 'S' contains a countable num of sample points then $X(s)$ will be a discrete rando

→ It has a countable number of distinct values.

ex- Toss a coin

↓

2 outcomes

S (Sample space) = {(Head) H, (Tale) T}

→ Now the random variable $X(s)$ will assign two distinct real numbers to these two values of the experiment.

* Continuous Random Variable

→ It is not limited to a finite number of distinct values.

→ It can have any value within a certain range.

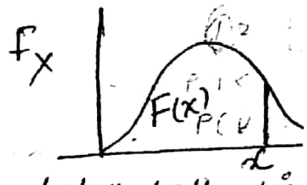
→ It has uncountable number of possible values.

Eg:

* Cumulative distribution Function (CDF)

→ The CDF of a random variable is defined as the probab that the random variable X takes place values less than or equal to x.

$$\boxed{\text{CDF}: F_X(x) = P(X \le x)}$$



x is dummy variable. CDF can be defined for both disc as well as continuous.

* Property 1 It is always bounded between 0 and 1.

$$\boxed{0 \le F_X(x) \le 1}$$

$F_X(\infty) = P(X \leq \infty$

(2) This property states that

$$\boxed{F_X(\infty) = 1}$$

Proof: $F_X(\infty) = P(X \leq \infty)$. The random variable $X \leq \infty$ becomes a certain event & has 100% probability.

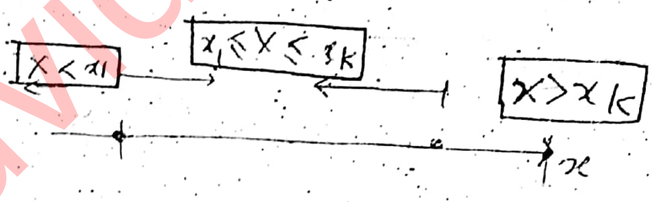
(3) $$\boxed{F_X(-\infty) = 0}$$

→ Acc to this $F_X(-\infty) = P(X \leq -\infty)$. The random variable X cannot have any value which is less than $-\infty$. Hence, $X \leq -\infty$ is a null event.

(4) $F_X(x)$ is a monotone non-decreasing fn.

$$\boxed{F_X(x_1) \leq F_X(x_2) \quad \text{for } x_1 < x_2}$$

→ **Calculate CDF**



(1) CDF for $X < x_1$

(2) CDF for $x_1 \leq X \leq x_k$

$$P(x_1 \leq X \leq x_k) = P(X = x_1) + P(X = x_2) + \dots P(X = x_k)$$
$$= \sum_{i=1}^{K} P(X = x_i)$$

(3) $X > x_k$

Then $(X \leq x_k)$ will include all the possible values from x_1 to x_k

Thus $P(X \leq x) = 1$ for $X > x_k$

← For complete Range

$$F_X(x) = \begin{cases} 0 & \text{for } X \leq x_1 \\ \sum_{i=1}^{K} P(X = x_i) & \text{for } x_1 \leq X \leq x_k \\ 1 & \text{for } X > x_k \end{cases}$$

Probability density function (PDF) $f_X(x)$

↳ describes a continuous random variable

PDF : $f_X(x) = \frac{d}{dx} F_X(x)$

Properties

① CDF can be derived from PDF by integrating

$$\boxed{F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx}$$

Proof as PDF is $f_X(x) = \frac{d}{dx} F_X(x)$

Integrating both sides

$$\int_{-\infty}^{x} f_X(x)\,dx = \int_{-\infty}^{x} \frac{d}{dx} F_X(x)$$

$$\int_{-\infty}^{x} f_X(x)\,dx = \Big[F_X(x)\Big]_{-\infty}^{x}$$

$$= \Big[F_X(x) - F_X(-\infty)\Big]$$

$$\boxed{\int_{-\infty}^{x} f_X(x)\,dx = F_X(x)} \qquad \text{as } F_X(-\infty) = 0$$

② PDF is a non-negative function for all values of x

$\boxed{f_X(x) \geq 0}$ for all x

As CDF is monotone increasing fn and PDF is derivative of monotone increasing fn

* ③ Area under PDF curve is unity

$$\boxed{\int_{-\infty}^{\infty} f_X(x)\,dx = 1}$$

Proof as $f_X(x) = \frac{d}{dx} F_X(x)$

Integrating both sides

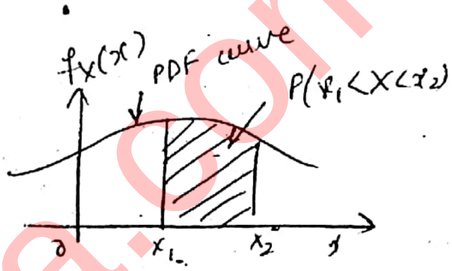
$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{\infty} \frac{d}{dx} F_X(x)\,dx$$

$$= \left[F_X(x)\right]_{-\infty}^{\infty}$$

$$= \left[F_X(\infty) - F_X(-\infty)\right]$$

$$= 1 - 0$$

$$\boxed{\int_{-\infty}^{\infty} f_X(x)\,dx = 1}$$



To find probability:

$$P(x_1 < X < x_2) = \int_{x_1}^{x_2} f_X(x)\,dx$$

→ Probability of obtaining X between x_1 and x_2 is equal to the area under PDF curve between values x_1 and x_2

* Joint CDF:

↳ Sometimes the outcome of an experiment requires more than one random variable to describe the experiment

$$F_{XY}(x,y) = P(X \le x, Y \le y)$$

gives the probability that the random variable X is less than or equal to specific value x and the random variable Y is less than or equal to a specific value y

① $F_{XY}(x,y)$ is a non-negative function ② Joint CDF is always continuous ③ It is a non-decreasing function of both x and y.

* Joint PDF

↳ $$f_{XY}(x,y) = \frac{\partial^2 [F_{XY}(x,y)]}{\partial x\,\partial y}$$

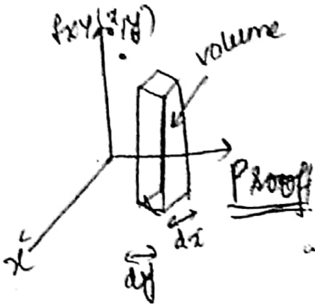
It is defined for two or more continuous random variables which may or may not be statistically independent.

It has 2 dummy variables x and y.

Properties of Joint PDF

(1) Joint PDF is always non-negative.
The derivative of a non-negative function must be non-negative is always!

(2) total volume under surface is



$$\boxed{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1}$$

comma represents AND

Proof: a) Joint PDF $= \frac{d^2}{dx\,dy}$ CDF $\leftarrow$ integrating both sides

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = \left[P(X \le x, Y \le y)\right]_{x=y=-\infty}^{x=y=\infty}$$

$$= P(X \le \infty, Y \le \infty) - P(X \le -\infty, Y \le -\infty)$$

$$= 1 - 0$$

$$= 1$$

(3) The probability of observing 'Y' in the interval from y_1 to y_2 and the probability of observing 'X' from x_1 to x_2 is given by the volume under surface represented by joint PDF

$$\boxed{P(x_1 < X \le x_2, y_1 < Y \le y_2) = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{XY}(x,y)\,dx\,dy}$$

Proof: We have

$$P(x_1 < X \le x_2, y_1 < Y \le y_2) = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{XY}(x,y)\,dx\,dy \quad -(1$$

$\rightarrow P(x_1 < X \le x_2) = \int_{x_1}^{x_2} f_X(x)\,dx$

similarly for Y

$\rightarrow P(y_1 < Y \le y_2) = \int_{y_1}^{y_2} f_Y(y)\,dy$

If two random variables X & Y are mutually in

$\rightarrow f_{XY}(x,y) = f_X(x)\,f_Y(y)$

substituting this in eqn (1)

1

$$P(x_1 < X \le x_2, y_1 < Y \le y_2) = \int_{y_1}^{y_2} f_Y(y)dy \int_{x_1}^{x_2} f_X(x)dx$$

$$\therefore \quad = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f_X(x)\,dx \cdot f_Y(y)\,dy$$

$$\boxed{P(x_1 < X \le x_2, y_1 < Y \le y_2) = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{XY}(x,y)\,dx \cdot dy}$$

* Conditional Probability density function -

If the random variables X and Y are not independent, then, the dependence of X on Y is expressed by the conditional PDF as

$$f_X(x|Y=y) = f_X(x/y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

where $f_Y(y)$ is marginal density.

This gives us conditional PDF of X given that $Y=y$

Properties:

1. This is non-negative fn.
2. Area under this is 1

$$\int_{-\infty}^{\infty} f_Y(y/x)\,dy = 1$$

$$\int_{-\infty}^{\infty} f_X(x/y)\,dx = 1$$

3. If x and y are independent events then

$$f_Y(y/x) = f_Y(y)$$

$$f_X(x/y) = f_X(x)$$

$$\Rightarrow f_{XY}(x,y) = f_X(x)\,f_Y(y)$$

* CDF, PDF

Que-1) $F_X(x) = \begin{cases} 0 & \text{for } x \le 0 \\ kx^2 & \text{for } 0 < x \le 10 \\ 100k & \text{for } x > 10 = 1 \end{cases}$

1) calculate $k = 0.01$
2) find $P(x \le 5)$ $P(0 < x \le 5) = F_X(5) - F_X(0) \rightarrow k x^2 \text{ } x=5$ and $P(5 < x \le 7)$ 0.24 $\Sigma(f_X 7) - \Sigma f_X($
   $\downarrow$ $f_X(7) - f_X(5)$
3) Plot PDF
   $\hookrightarrow d/dx \, F(X)$

Que-2) PDF is $f_X(x) = \begin{cases} k, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$

where k is a constant $\int_{-\infty}^{\infty} f_X(x) = 1$ $k = \frac{1}{b-a}$

(1) Determine the value of k
(2) let $a = -1, b = 2$ calculate $P(|X| \le c)$ for $c = 1/2$
$\int_{-1/2}^{1/2} \frac{1}{b-a} dx = 1/3$

Joint PDF

Que- 3) $f_{XY}(x,y) = \frac{1}{4} e^{-|x|-|y|}$ $-\infty < x < \infty, \; -\infty < y < \infty$

Determine

(1) Whether Random variables X and Y are statistically independent
(2) Probability that $X \le 1$ and $Y \le 0$
$\hookrightarrow \frac{1}{4}(2 - e^{-1})$

$\int_{-\infty}^{1} \int_{-\infty}^{0} f_{XY}(x,y) \, dx \, dy$

Que - 4) The Joint PDF is

$f_{XY}(x,y) = C \cdot e^{-(ax+by)} u(x) \cdot u(y)$
$c = ab$

$F_{XY} = \int\int f_{XY}(x,y)$ Find C.

Marginal densities

The individual probability densities $f_X(x)$ and $f_Y(y)$ can be obtained from joint PDF $f_{XY}(x,y)$. Then these individual densities are called as marginal densities or marginal PDFs.

→ Expression:-

CDF is given by $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

similarly joint CDF of 2 random variables X & Y is

$$F_{XY}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(x,y)\,dx\,dy$$

If we want to obtain CDF of a single random variable X

i.e. $F_X(x) = P(X \le x)$

Then $F_{XY}(x,y)$ becomes (Y can have any value from $-\infty$ to $+\infty$)

$$F_{XY}(x,\infty) = P(X \le x, -\infty < Y \le \infty)$$

$$= \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy$$

Now as $f_{XY}(x,y) = \frac{d}{dx} F_X(x)$

we have

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx}\left[\int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy\right]$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy.$$

Hence this is the marginal density and marginalised PDF can be obtained by integrating the joint PDF with respect to the undesired random variable 'Y'.

* Statistical averages :-

↳ PDF provides more info about a random variable than needed, so it is simpler to describe a random variable by a few characteristics numbers.

1. Mean / average / Expected value $\bar{X}$

→ Let us consider a discrete random variable X which has the possible values of $x_1, x_2 \ldots$ with probabilities of occurence to be $P(x_1), P(x_2) \ldots$.

→ N independent observations of random variable X, then we expect that outcome $X = x_1$ will occur $N P(x_1)$ times, the outcome $X = x_2$ would occur $N P(x_2)$ times etc.

→ The arithmetic sum of independent observations will be :-

$$= x_1 P(x_1) \cdot N + x_2 \cdot P(x_2) \cdot N + x_3 \cdot P(x_3) \cdot N + \ldots$$

$$= N \sum_{i=1}^{K} x_i P(x_i)$$

The Mean is $\boxed{\bar{X} = \frac{N}{N} \sum_{i=1}^{K} x_i P(x_i) = m_x = E[X]}$

* Mean Value of a continuous Random variable

→ If the random variable 'X' is continuous random variable then the value x_1, x_2, x_3 comes very close to each other so the summation sign gets converted to integration sign. The integration is carried out from $-\infty$ to ∞.

$$\boxed{P(x_1 < X \le x_2) = \int_{x_1}^{x_2} \underset{\text{PDF}}{f_X(x)}\, dx}$$

Put $x_2 = x_1 + dx$

$f_x = PD$

$$P(x_1 < X \le x_2) = f_X(x)\, dx$$

- dx is very small that $f_X(x)$ can be assumed to be constant in the range x_1 to (x_1+dx). Putting x in place of x_1.

$$P(x < X \leq x+dx) = f_X(x)\,dx$$

- This is the probability of observing X in very small range.

$$P(x < X \leq x+dx) = P(x)$$

So

$$\boxed{P(x) = f_X(x)\,dx}$$

→ Now, put value of $P(x)$ in the original eqn

$$m_x = \int_{i=1}^{R} x_i\, f_X(x)\,dx$$

Replace summation by integration

$$\boxed{m_x = \int_{-\infty}^{\infty} x\, f_X(x)\,dx}$$

- If function $g(x)$ transforms X into some other random variable

$$\boxed{E[g(X)] = \int_{-\infty}^{\infty} g(X)\, f_X(x)\,dx}$$

* <u>nth moment of Random variable X.</u>

It is <u>mean value of X^n</u>. Thus the expression for nth moment is

$$n^{th} \text{ moment of } X = E[X^n] = \overline{X^n}$$
$$= \int_{-\infty}^{\infty} X^n f_X(x)\,dx$$

* 1st moment of X (Mean value)

Put $n=1$

$$E[X] = \bar{X} = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = m_x$$

This is the same as average value of X.

* 2nd moment of X (Mean square value).

Put $n=2$

$$E[X^2] = \overline{X^2} = \int_{-\infty}^{\infty} x^2 f_X(x) dx$$

* Central Moment.

↳ Expected value of the difference b/w random variable X and its mean value m_x.

$$n^{th} \text{ central moment of } X = \boxed{E[(X-m_x)^n] = \int_{-\infty}^{\infty} (x-m_x)^n f_X(x).}$$

* 2nd central moment (Variance) σ_x^2

$$\text{Variance of } X = E[(X-m_x)^2] = \sigma_x^2$$

$$= \int_{-\infty}^{\infty} (x-m_x)^2 f_X(x) dx$$

$$E[X^2 - 2Xm_x + m_x^2] = E[X^2] - 2m_x E[X] + m_x^2$$

$$= E[X^2] - 2m_x \cdot m_x + m_x^2$$

$$\boxed{\sigma_x^2 = E[X^2] - m_x^2}$$

Variance = Mean square value − square of the Mean.

Ex ⟹ $f_X(x) = \frac{1}{2} e^{-|x|}$ for $-\infty < x < \infty$ { $m_X = 0$, Mean = ... }

Standard deviation (σ_X)

$$S.D = \sqrt{\text{variance}}$$

So $\sigma_X = \sqrt{E[X^2] - m_X^2}$

- This gives the width of its PDF.

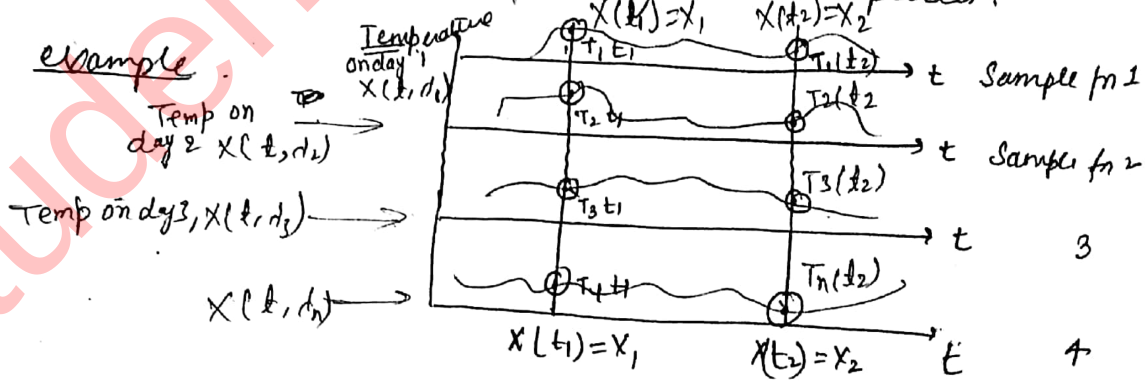
* Classification of process (Random)

Stationary SSS, Wide sense, Ergodic process

* **Stationary and Non-stationary Random processes.**

↳ A random process whose statistical chrt do not change with time is known as stationary random process. Hence the shift of time origin will not have any effect on the stationary random process.

example



→ Let us consider a random process X(t) which is initiated at $t = -\infty$, let $X(t_1), X(t_2) \cdots X(t_n)$ denote random variable obtained by observing random process X(t) at instants $t_1, t_2 \ldots t_n$.

Hence Joint CDF is $F_{X(t_1), X(t_2) \cdots X(t_n)}(x_1, x_2 \ldots x_n)$

or

$F_{X_1, X_2}(x_1, x_2 \ldots x_n)$

* Random Process

↳ Properties of Random Signal :-

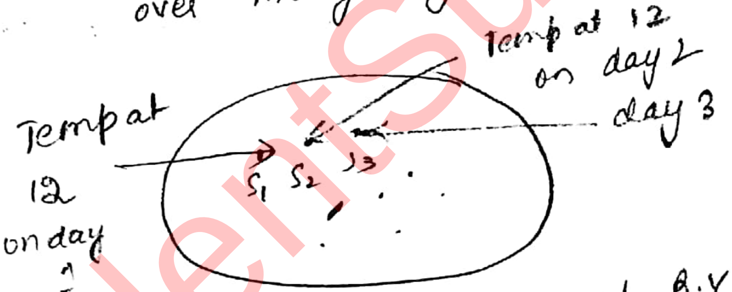
(i) They are functions of time and defined over some time interval.

(ii) It is not possible to describe exactly the waveform of this signal with respect to time.

Random Process / Stochastic Process

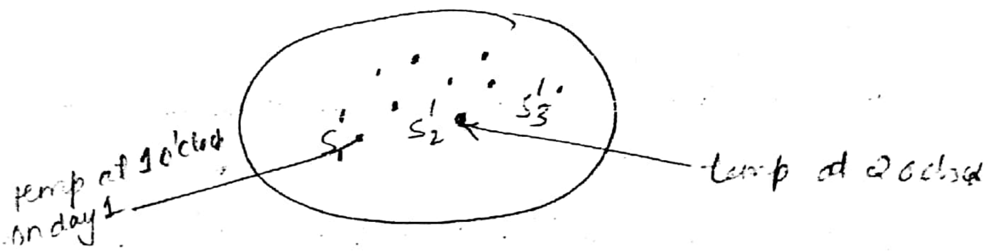
The notion of a random ~~variable~~ process is an extension of the random variable (R.V.)

ex- let us consider a random variable X which represents the temperature of a city at 12 o'clock. The temperature X is a random variable and can take on different values every day. To get the complete idea about the random variable X, we will have to record values of X at 12 o'clock over many days.

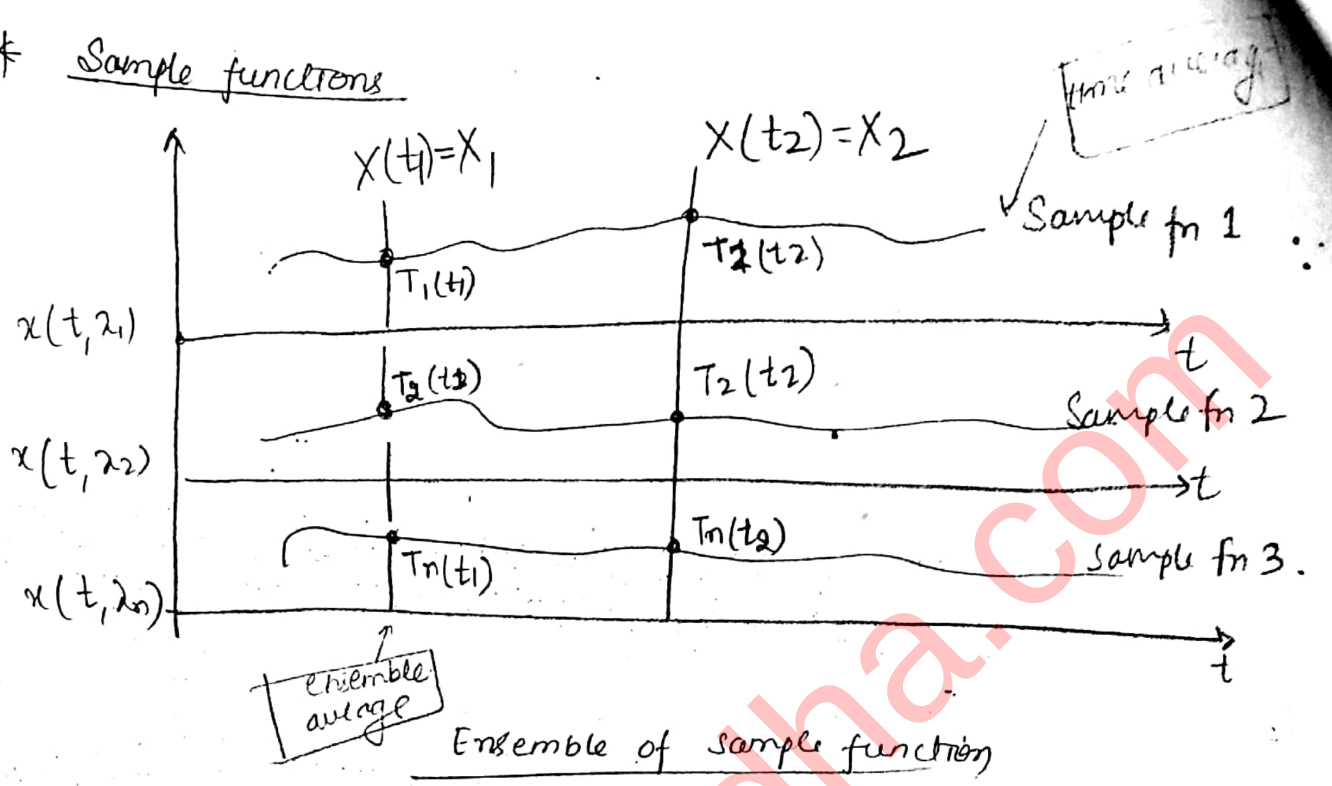


Sample space of R.V X at 12 o'clock

But the temperature is also a function of time. Hence temperature at 1 o'clock will have different distribution from temperature at 12 o'clock. So, we define another random variable to represent city temperature at 1 o'clock.



* Sample functions



Ensemble of sample function

Ensemble : collection of all possible sample function is called as an ensemble.

Random Process : The ensemble comprised of functions of time is called as Random/ stochastic process.

We can also define the random process as ensemble of random variables which are functions of time and hence denote the random process as $X(t)$

We specify a random variable X by repeating an experiment a large number of times and from the outcomes of the experiment, we determine $f_X(x)$. We do the same thing at each value of t, to specify a random process $X(t)$.

* Ensemble Mean or Ensemble Average

↳ These are taken over the ensemble of waveforms at a fixed instant of time.

∴ ex - Ensemble mean value taken at $t = t_1$ will contain all the values taken at $t = t_1$ it means $T_1(t_1)$, $T_2(t_1)$, $T_3(t_1)$ ... $T_n(t_1)$

$$\overline{X(t)} = \int_{-\infty}^{\infty} X f_X(x;t)\, dx$$

t is treated as constant as time is fixed.

Ensemble mean is a function of time

* Averages

↳ The ensemble averages are obtained at constant values of t, whereas time averages are obtained by changing time. t.

$$\boxed{\text{Time average} = m_X(T) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t)\, dt}$$

(Here we have time 't' treated as variable)

↳ The autocorrelation function for a random process X(t) is defined as

$$\boxed{R_X(t_1, t_2) = E[X(t_1) X(t_2)]}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X_1 X_2}(x_1, x_2)\, dx_1 dx_2$$

→ The autocorrelation function indicates the similarity between the amplitudes $X(t_1)$ and $X(t_2)$ of the random process $X(t)$ at time instants t_1 and t_2.

→ The value of $R_X(t, t_2)$ is obtained by taking product of the values of the sample functions at instants t_1 and t_2 and then taking the mean of this product.

* Auto correlation function of a Random Process

This provides the spectral information about the random process. It is defined as :

$$R_X(\tau) = E[X(t+\tau)\, X(t)]$$

Properties :-

① The mean square value of a process is the value of autocorrelation at $\tau = 0$. This means

$$R_X(0) = E[x^2(t)]$$

② It states that autocorrelation function $R_X(\tau)$ is an even function of τ. This means

$$R_X(\tau) = R_X(-\tau)$$

* Proof :

$$R_X(\tau) = E[X(t+\tau)\, X(t)]$$

substitute $t = t - \tau$

$$R_X(-\tau) = E[X(t)\, X(t-\tau)]$$

③ It states that the autocorrelation function $R_X(\tau)$ is maximum at $\tau = 0$

$$|R_X(\tau)| \leq R_X(0)$$

* Classification of Random Processes
- Stationary
- Wide sense stationary
- Ergodic

A random process whose statistical characteristics do not change with time is known as stationary Random process. Hence shift of time origin will not have any effect on the stationary random process.

→ Let us consider a process $X(t)$ initiated at $t = -\infty$
let $X(t_1), X(t_2) \ldots X(t_n)$ denote random variable obtained by observing the random process $X(t)$ at instants $t_1, t_2 \ldots t_n$.

Then its CDF is denoted by

$$F_{X(t_1), X(t_2), X(t_n)}(x_1, x_2 \ldots x_n)$$

Strict sense stationary

→ If Joint CDF of the original set of random variable is equal to that of the new set of random variables obtained after a time shift τ.

$$F_{X(t_1), X(t_2), X(t_3)}(x_1, x_2 \ldots x_n) = F_{X(t_1+\tau) X(t_2+\tau) \ldots}(x_1, x_2 \ldots x_n)$$

for all time shifts τ, all n and all possible choices of $t_1, t_2 \ldots t_n$.

* Jointly Stationary Process

The two stationary processes $X(t)$ and $Y(t)$ are initiated at $t = -\infty$ are called as jointly stationary if the joint distribution function of the random variables $X(t_1), X(t_2) \dots X(t_n)$ and $Y(t_1'), Y(t_2') \dots Y(t_j')$ are invarient with respect to the location of origin $t=0$ for all n and j and for all the choices of $t_1, t_2, \dots t_n$ and $t_1', t_2', \dots t_j'$.

* Mean, Corelation & covariance functions of a Stationary Random Process

① Mean

Let us consider a stationary random process $X(t)$. The mean value of such a random process is given by :-

$$m_X(t) = \text{Expected value of } x(t)$$

$$= E\left[X(t)\right]$$

$$= \int_{-\infty}^{\infty} x \underbrace{f_{X(t)}(x)\, dx}_{\text{First order PDF of } X(t)}$$

considering the first property of stationary processes we can say that $f_{X(t)} x$ is independent of time. Hence,

$$m_X(t) = m_X$$

② Autocorelation of $X(t)$

It is defined as the expectation of the product of two random variables $X(t_1)$ and $X(t_2)$

$$R_X(t_1, t_2) = E\left[X(t_1) X(t_2)\right]$$

$$R_X(t_1,t_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2\, f_{X(t_1)} f_{X(t_2)}(x_1,x_2)\, dx_1 dx_2$$

→ From the 2nd property of stationary random process, we can say that for stationary random process, $f_{X(t_1)} f_{X(t_2)}(x_1,x_2)$ depend only on the difference between observation times t_1 and t_2

→ Hence autocorrelation fn. depends only on the time difference (t_2-t_1).

$$R_X(t_1,t_2)=R_X(t_2-t_1)$$

③ Autocovariance function of a stationary Process

↳ It is given as :-

$$C_X(t_2-t_1) = E[(X(t_1)-m_X)(X(t_2)-m_X)]$$
$$= R_X(t_2-t_1) - m_X^2$$

↳ Hence autocovariance function of a stationary process $X(t)$ is dependent only on the time difference. It is possible to calculate the autocovariance if the mean and autocorrelation of the random process are known.

* Wide sense stationary Process :

A process may not be stationary in strict sense, still it may have mean value $(m_X(t))$ and ~~the~~ an autocorrelation function which are independent of the shift of time origin. This means

$$m_X(t) = \text{constant}$$
$$R_X(t_1,t_2) = R_X(t_2-t_1)$$

All the stationary processes are wide-sense-stationary but every wide-sense stationary process may not be strictly stationary.

* Ergodicity (Ergodic process)

Let us take a sample function $x(t)$ of wide sense stationary process $X(t)$. Let the observation interval be $-T \leq t \leq T$. Then, the average (Time average) is

Time average :

$$m_X(T) = \frac{1}{2T} \int_{-T}^{T} x(t)\, dt$$

Mean value of the time average $m_X(T)$ is

$$E[m_X(T)] = E\left[\frac{1}{2T} \int_{-T}^{T} x(t)\, dt\right]$$

Interchanging the operation of expectation & integration

$$E[m_X(T)] = \underbrace{\frac{1}{2T} \int_{-T}^{T} \underbrace{E[x(t)]}_{\text{Mean value of } x(t) = X(t)}\, dt}_{\text{Mean of Random process } X(t) = m_X}$$

m_X ↓ ensemble average.

$$E[m_X(T)] = m_X$$

Conditions for Ergodicity

(1) The time average $m_X(T)$ approaches the ensemble average m_X (with observation interval T tending to ∞)

$$\lim_{T \to \infty} \underbrace{m_X(T)}_{\text{Time aver}} = \underbrace{m_X}_{\text{ensemble average}}$$

② The variance of $m_X(T)$ which is treated as a random variable, approaches zero with observation interval $T \to \infty$

$$\therefore \lim_{T \to \infty} \text{var}[m_X(T)] = 0$$

* Time averaged auto-correlation function : $R_X(\tau, T)$

It is defined for the sample function $x(t)$

$$R_X(\tau, T) = \frac{1}{2T} \int_{-T}^{T} x(t+\tau)\, x(t)\, dt$$

conditions for ergodicity

The process $X(t)$ is said to be ergodic in autocorrelation function if the following conditions are satisfied :-

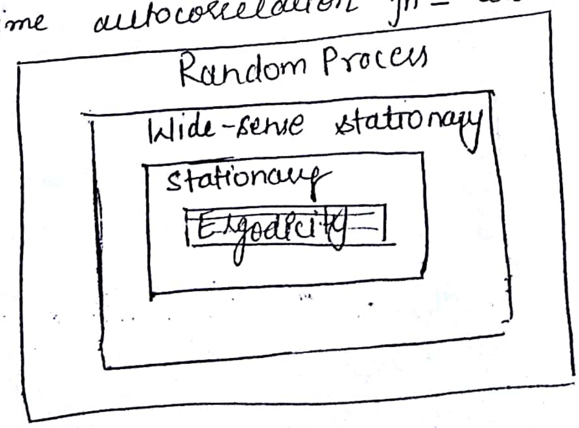
(i) $\lim_{T \to \infty} R_X(\tau, T) = R_X(\tau)$.

This means that time averaged autocorrelation function approaches the autocorrelation fn of the system with observation interval $T \to \infty$.

(ii) $\lim_{T \to \infty} \text{var}[R_X(\tau, T)] = 0$

This means that the variance of $R_X(\tau, T)$ which is a random variable, approaches zero with the observation interval tending to ∞.

Ergodicity
- Time average = Ensemble average
- Time autocorrelation fn = autocorrelation fn.



Correlation b/w Random variables:-

Let there be 2 outcomes described by random variables X and Y. Several trials are carried out.

The covariance is given by

$$\sigma_{XY} = E[(X-m_x)(Y-m_y)]$$
$$= E[XY - X\cdot m_y - m_x\cdot Y + m_x\cdot m_y]$$
$$= E[XY] - E[X]\cdot m_y - m_x\cdot E[Y] + m_x\cdot m_y$$
$$= E[XY] - m_x\cdot m_y - m_x\cdot m_y + m_x\cdot m_y$$

$$\boxed{\sigma_{XY} = E[XY] - m_x\cdot m_y}$$

If $E[XY] = m_x\cdot m_y$ then

$$\sigma_{XY} = 0$$

* **Correlation coefficient:** for comparison of correlations, we use correlation coefficient

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

If X and Y are uncorrelated then $\rho_{XY} = 0$

$$-1 < \rho_{XY} < 1$$

Central limit theorem

It is used when PDF of several independent random variables is to be obtained.

* → under certain conditions, the PDF of sum of large no. of independent random variables tend to approach Gaussian PDF, independent of probability densities of the variables.

Proof :-

→ Let X and Y be two independent variables. Let their addition be equal to another random variable Z i.e

$$X + Y = Z$$

→ Let PDF of Z be $f_Z(z)$. and $Y = Z - X$ (regardless of value of X)

or CDF → The probability that $Z \leq z$ is denoted by $P(Z \leq z)$ and is given by

$$P(Z \leq z) = P[(X+Y) \leq z]$$

The event $Z \leq z$ is a joint event [$y \leq z-x$ and x to have any value in range $-\infty, \infty$]

* CDF →
$$F_Z(z) = P(Z \leq z) = P[X \leq \infty, Y \leq (z-x)] \quad -(1)$$

Joint CDF can be expressed as (in terms of PDF)

$$\rightarrow P(x_1 < X < x_2, y_1 < Y < y_2) = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{XY}(x,y)\,dx\,dy \quad -(2)$$

using this equation in 1, we have

CDF of Z

$$\boxed{F_Z(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{(z-x)} f_{XY}(x, z-x)\,dx\,dy} \quad -(3)$$

* Note → PDF of Z is

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

$$= \int_{-\infty}^{\infty} f_{XY}(x, z-x)\,dx \cdot \frac{d}{dz}\left[\int_{-\infty}^{z-x} dy\right]$$

→ But as X and Y are independent variables and
$y = z - x$

Therefore

$$dy = dz$$

→ Now the equation becomes

$$\boxed{f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z-x)\,dx \cdot 1} \quad -④$$

case If X and Y are independent variables, then PDF is given by

$$\rightarrow f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$$

so

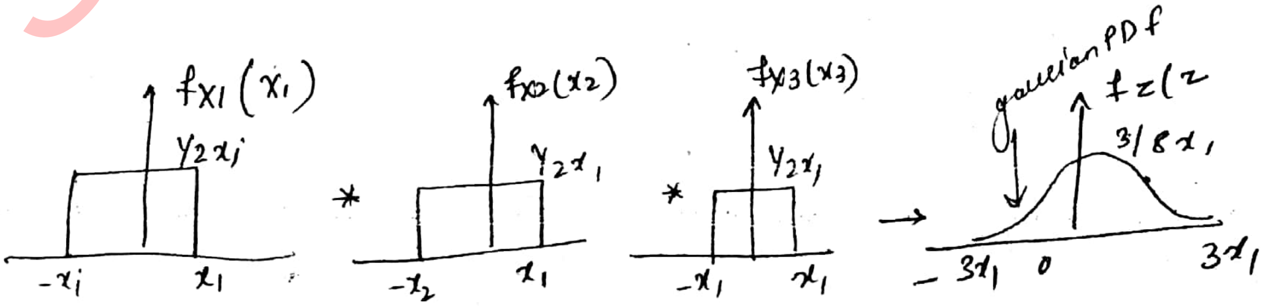
$$\rightarrow f_{XY}(x, z-x) = f_X(x) \cdot f_Y(z-x) \quad - ⑤$$

Putting this value in equation ④

$$f_Z(z) = \underbrace{\int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x) \cdot dx}$$

It is representing convolution of PDF of X and Y hence

$$\boxed{f_Z(z) = f_X(x) * f_Y(y)}$$



$X(t) \longrightarrow$ [Impulse Response $h(t)$] $\longrightarrow Y(t)$

* Power Spectral density (Frequency domain)

Let the impulse response of a LTI filter is equal to the IFT of the transfer function $H(f)$.

$\therefore$

$$h(\tau_1) = \int_{-\infty}^{\infty} H(f) \cdot e^{j2\pi f \tau_1} \, df$$

The mean square value of $Y(t)$ is given by :- Put here

$$E[Y^2(t)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) R_X(\tau_2 - \tau_1) \, d\tau_1 \, d\tau_2$$

Putting value of $h(\tau_1)$ in this equation

$$E[Y^2(t)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} H(f) \cdot e^{j2\pi f \tau_1} \, df \right] h(\tau_2) R_X(\tau_2 - \tau_1) \, d\tau_1 \, d\tau_2$$

Rearranging we have

$$E[Y^2(t)] = \int_{-\infty}^{\infty} df \, H(f) \int_{-\infty}^{\infty} d\tau_2 \, h(\tau_2) \int_{-\infty}^{\infty} R_X(\tau_2 - \tau_1) e^{j2\pi f \tau_1} \, d\tau$$

let $\tau = (\tau_2 - \tau_1) \longrightarrow \tau_1 = \tau_2 - \tau$

$d\tau_1 = -d\tau$

Now the equation becomes :-

$$E[Y^2(t)] = \int_{-\infty}^{\infty} df \, H(f) \int_{-\infty}^{\infty} d\tau_2 \, h(\tau_2) \int_{-\infty}^{\infty} R_X(\tau) \cdot e^{j2\pi f(\tau_2 - \tau)} \, d\tau$$

$$= \int_{-\infty}^{\infty} df \, H(f) \int_{-\infty}^{\infty} d\tau_2 \, h(\tau_2) \, e^{j2\pi f \tau_2} \int_{-\infty}^{\infty} R_X(\tau) \cdot e^{-j2\pi f \tau} \, d\tau$$

$$\boxed{\text{But } \int_{-\infty}^{\infty} h(\tau_2) \cdot e^{j2\pi f \tau_2} \, d\tau_2 = H^*(f)}$$

$$E[Y^2(t)] = \int_{-\infty}^{\infty} df \cdot H(f) \cdot H^*(f) \cdot \int_{-\infty}^{\infty} R_X(\tau) \, e^{-j2\pi f \tau} \, d\tau$$

$$H(f) \cdot H^*(f) = |H(f)|^2$$

Therefore $E[Y^2(t)] = \int_{-\infty}^{\infty} df \, |H(f)|^2 \underbrace{\int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f \tau} d\tau}_{\text{fourier transform of autocorrelation fn } R_X(\tau).}$

$$x(t) \xleftrightarrow{F} \int_{-\infty}^{\infty} R_X(\tau) \cdot e^{-j2\pi f \cdot \tau} d\tau$$

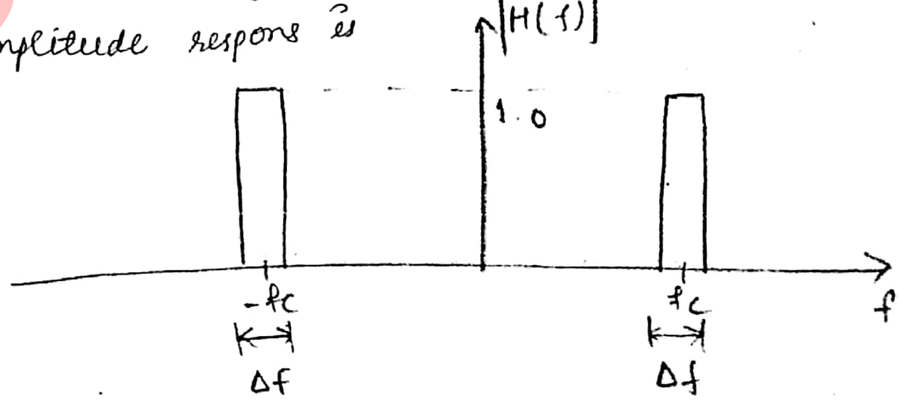
change this by a new parameter $S_X(f)$
Power spectrum \ Power spectral density

$$E[Y^2(t)] = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f) df$$

So mean of square value of the output $E[Y^2(t)]$ of a LTI filter for a wide sense stationary process $X(t)$ applied at its input, is equal to integration of psd and squared magnitude of transfer function over the entire frequency range $-\infty$ to $+\infty$.

* Physical Significance of PSD

Let us assume that the random process $X(t)$ is passed through an ideal narrowband filter whose amplitude response is



$$|H(f)| = \begin{cases} 1 & \text{for } |f \pm f_c| < \frac{1}{2}\Delta f \\ 0 & \text{for } |f \pm f_c| > \frac{1}{2}\Delta f \end{cases}$$

→ where Δf represents bandwidth of the filter.

→ Now

$$E[Y^2(t)] = \int_{-\infty}^{\infty} |H(f)|^2 \cdot S_X(f)\, df$$

→ We assume that Δf is very small as compared to f_c and $S_X(f)$ is a continuous function of frequency. Hence

$$E[Y^2(t)] = \int_{f_c - \frac{1}{2}\Delta f}^{f_c + \frac{1}{2}\Delta f} |H(f)|^2 \, S_X(f)\, df$$

$$\approx 2\Delta f \, S_X(f_c)$$

→ $S_X(f)$ - represents the frequency density of the average power in the random process $X(t)$ calculated at $f = f_c$. The dimensions of psd are watt/Hz.

* Properties of PSD

Einstein - Wiener - Khintchine relations :-

The psd $S_X(f)$ and autocorrelation fn $R_X(\tau)$ of a wide-sense stationary process $X(t)$ always forms a Fourier transform pair as under :-

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \cdot e^{-j2\pi f\tau} \cdot d\tau$$

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(\tau) \cdot e^{-j2\pi f\tau} d\tau$$

These expressions are known as Einstein - Wiener - Khintchine relation These relations indicate that

$$S_X(f) \xleftrightarrow{F} R_X(\tau)$$

Hence if one is known we can find other one.

* Property 1

$$S_X(0) = \int_{-\infty}^{\infty} R_X(\tau)\, d\tau$$

The value of psd of a wide sense stationary process at $f=0$ is equal to the total area under the curve of autocorrelation function. This means

Proof: We know that

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \cdot e^{-j2\pi f \tau} d\tau$$

Substituting $f=0$

$$S_X(0) = \int_{-\infty}^{\infty} R_X(\tau) \cdot e^{0}\, d\tau$$

$$= \int_{-\infty}^{\infty} R_X(\tau) \cdot d\tau$$

* Property 2

The mean square value of a WSS random process is equal to the total area under the curve of power spectral density. This means that

$$E\left[x^2(t)\right] = \int_{-\infty}^{\infty} S_X(f)\, df$$

Proof: We know

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) \cdot e^{j2\pi f \tau} df$$

Put $\tau = 0$

$$R_X(0) = \int_{-\infty}^{\infty} S_X(f) \cdot e^{0} df = \int_{-\infty}^{\infty} S_X(f) df$$

and $E[X^2(t)] = R_X(0)$

Hence $E[X^2(t)] = \int_{-\infty}^{\infty} S_X(f) df$

Property 3 : PSD of a WSS will always be non-negative for all the values of f. This means

$$S_X(f) \geq 0 \text{ for all } f.$$

Proof:

PSD $S_X(f) = \dfrac{E[Y^2(t)]}{2\Delta f}$

As the mean square value $E[Y^2(t)]$ will always be non-negative, Δf is also non-negative. Hence $S_X(f)$ is also non-negative.

Property 4 : PSD of a real valued Random process is an even function of frequency.

$$S_X(-f) = S_X(f)$$