B.E.

Third Semester Examination, May-2008

MATHEMATICS-III

Note : Attempt any five questions.

Part-A

Q. 1. (a) Obtain a Fourier series to represent e^{-ax} from $x = -\pi$ to $x = \pi$. Hence derive series for $\frac{\pi}{\sinh \pi}$.

Ans. Let $$e^{-ax} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

According to Euler's formulae,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\,dx = \frac{1}{\pi}\left[\frac{e^{-ax}}{-a}\right]_{-\pi}^{\pi} = -\frac{1}{a\pi}\left(e^{-a\pi} - e^{a\pi}\right) = \frac{2\sinh a\pi}{a\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\,dx = \frac{1}{\pi}\left[e^{-ax}\frac{\sin nx}{n} - \int_{-\pi}^{\pi} e^{-ax}(-a)\frac{\sin nx}{n}dx\right]$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}(-a\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\left(-a(1-)^n\right) - \frac{e^{a\pi}}{a^2+n^2}\left(-a(-1)\right)^n\right]$$

$$= \frac{(-1)^n a.2\sinh a\pi}{\left(a^2+n^2\right)\pi}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\sin nx\,dx = \frac{1}{\pi}\left[e^{-ax}\left(\frac{-\cos nx}{n}\right) - \int_{-\pi}^{\pi} e^{-ax}(-a)\left(\frac{-\cos n\pi}{n}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}(a\sin nx + n\cos nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\left(n(-1)^n\right) - \frac{e^{-ax}}{a^2+n^2}\left(n(-1)^n\right)\right]$$

$$b_n = \frac{2n(-1)^n \sinh a\pi}{\pi\left(a^2+n^2\right)}$$

Putting the value of a_0, a_n & b_n we get,

$$e^{-ax} = \frac{2\sinh a\pi}{\pi}\left[\frac{1}{2a} - a\left(\frac{\cos x}{a^2+1^2} - \frac{\cos 2x}{a^2+2^2} + \ldots\ldots\right) - \left(\frac{\sin x}{a^2+1^2} - \frac{2\sin 2x}{a^2+2^2} + \ldots\right)\right]$$

Put x = 0, a = 1

$$1 = \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} - \left(\frac{1}{2} - \frac{1}{2^2+1} + \frac{1}{3^2+1}\ldots\ldots\ldots\ldots\ldots\ldots\right)\right]$$

$\Rightarrow$

$$\frac{\pi}{\sinh\pi} = 2\left[\frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1}\ldots\ldots\ldots\right].$$

Q. 1. (b) Obtain a half range cosine series for

$$f(x) = \begin{cases} kx & \text{for } 0 \le x \le \frac{1}{2} \\ k(1-x) & \text{for } \frac{1}{2} \le x \le 1 \end{cases}$$

and hence find sum of the series

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.$$

Ans. $\quad f(x) = kx \qquad 0 \le x \le 1/2$

$\quad = k(1-x) \qquad 1/2k \le x \le 1$

f(x) may be consider as even function in $(-l,l)$ by extending function in (–l, 0) in such a way that it become symmetrical about y-axis.

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{l}\right)$$

By Euler's formula,

$$a_0=\frac{2}{l}\int_0^l f(x)dx=\frac{2}{l}\left[\int_0^{l/2}kx\,dx+\int_{l/2}^{l}k(l-x)dx\right]$$

$$=\frac{2}{l}\left[\left(\frac{kx^2}{2}\right)_0^{l/2}+\left\{k\left(lx-\frac{x^2}{2}\right)\right\}_{l/2}^{l}\right]^{l/2}$$

$$=\frac{2}{l}\left[\frac{kl^2}{8}+k\left(l^2-\frac{l^2}{2}\right)-k\left(\frac{l^2}{2}-\frac{l^2}{8}\right)\right]=\frac{kl}{2}$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx=\frac{2}{l}\left[\int_0^{l/2}kx\cos\frac{n\pi x}{l}dx+\int_{l/2}^{l}k(l-x)\cos\frac{n\pi x}{l}dx\right]$$

$$=\frac{2}{l}\left[k\left\{x\frac{\sin\frac{n\pi x}{l}}{n\pi/l}+\frac{\cos\frac{n\pi x}{l}}{(n\pi/l)^2}\right\}_0^{l/2}+k\left\{(l-x)\frac{\sin\frac{n\pi x x}{l}}{n\pi/l}-\frac{\cos n\pi-\cos n\pi/2}{(n\pi/l)^2}\right\}\right]$$

$$=\frac{2}{l}\left[k\left(\frac{l}{2}\frac{\frac{\sin n\pi}{2}}{n\pi/l}+\frac{\cos n\pi/2}{(n\pi/l)^2}-\frac{l}{(n\pi/2)}\right)+k\left(-\frac{l}{2}\frac{\sin n\pi/2}{n\pi/l}-\frac{\cos n\pi-\cos n\pi/2}{(n\pi/l)^2}\right)\right]$$

$$=\frac{2kl}{n^2\pi^2}\left[2\cos\frac{n\pi}{2}-1-(-1)^n\right]$$

$$a_1=0,\ a_2=\frac{2kl}{2^2\pi^2}(-2-1-1)=-\frac{2kl}{\pi^2}$$

$$a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ a_6 = \frac{2kl}{(6)^2 \pi^2}(-2-1-1) = \frac{2kl}{3^2 \pi^2}$$

$$\therefore \quad f(x) = \frac{kl}{4} - \frac{2kl}{\pi^2}\left[\frac{\cos 2x\frac{\pi}{l}}{1^2} + \frac{\cos 6x\pi/l}{3^2} + \frac{\cos 10x\pi/l}{5^2} + \ldots\ldots\right]$$

Put $x = 0$

$$f(x) = \frac{kl}{4} - \frac{2kl}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$$

Or
$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}.$$

Q. 2. (a) Express the function :

$$f(x) = \begin{cases} 1 & \text{for} \quad |x| \le 1 \\ 0 & \text{for} \quad |x| > 1 \end{cases}$$

as a Fourier integral.

Hence evaluate $\displaystyle\int_0^\infty \frac{\sin\lambda \cos\lambda x}{\lambda} d\lambda$.

Ans.
$$f(x) = \begin{cases} 1 & \text{for} \quad |x| \le 1 \\ 0 & \text{for} \quad |x| > 1 \end{cases}$$

The Fourier integral for

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos\lambda(t-x)\,dt\,d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \int_{-1}^1 1 - \cos\lambda(t-x)\,dt\,d\lambda = \frac{1}{\pi}\int_0^\infty \left[\frac{\sin\lambda(t-x)}{\lambda}\right]_{-1}^1 d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin\lambda(1-x) - \sin\lambda(-1-x)}{\lambda} d\lambda$$

$$=\frac{1}{\pi}\int_0^\infty \frac{\sin\lambda(1-x)+\sin\lambda(1+x)}{\lambda}d\lambda=\frac{2}{\pi}\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}d\lambda$$

$$\therefore \qquad \int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}d\lambda=\frac{1}{2}\pi f(x)$$

$$=\begin{cases}\frac{1}{2}\pi & |x|\le 1\\ 0 & |x|>1\end{cases}$$

at $x=\pm 1$, the function is discontinuous & hence integral has the value $\frac{1}{2}\left(\frac{\pi}{2}+0\right)=\frac{\pi}{4}$.

Q. 2. (b) State and prove convolution theorem for Fourier transforms.

Ans. Fourier transform of convolution theorem :

The convolution of two function $f(x)$ & $g(x)$ over the interval $(-\infty,\infty)$ is defined as,

$$f*g=\int_{-\infty}^{\infty} f(u)g(x-u)\,du=h(x)$$

Proof : The Fourier transform of convolution of $f(x)$ & $g(x)$ is product of their Fourier transform i.e.,

$$F(f(x)*g(x))=F(f(x)).F[g(x)]$$

We have,

$$F\{f(x)*g(x)\}=F\left[\int_{-\infty}^{\infty} f(u)g(x-v)\,du\right]$$

$$=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x)g(x-u)du\right]e^{-sx}dx=\int_{-\infty}^{\infty} f(x)\left[\int_{-\infty}^{\infty} g(x-u).e^{isx}dx\right]du$$

$$=\int_{-\infty}^{\infty} f(x)\left[\int_{-\infty}^{\infty} e^{is(x-u)}.g(x-u)d(x-u)\right]e^{isu}\,du$$

$$=\int_{-\infty}^{\infty} e^{-isu}f(x)\left[\int_{-\infty}^{\infty} e^{ist}g(t)\,dt\right]du \text{ when } x-u=j$$

$$=\int_{-\infty}^{\infty} e^{isu}f(u)\,du.F(g(t))$$

$$= \int_{-\infty}^{\infty} e^{isx} f(x)dx. F[g(x)]$$

$$= F[f(x)].F[g(x)].$$

Part-B

Q. 3. (a) Define analytic functions. State and prove the C-R equations for an analytic function.

Ans. Analytic function :

Consider a single valued function f(z) in a domain D. The function $f(z)$ is said to be analytic at point z = a if there exists a neighbourhood $|z-a|<\delta$ at all points of which $f'(z)$ exists.

If $f'(z)$ exists at every point of a domain, then we can say that the function is analytic or regular analytic in the domain.

Necessary and sufficient conditions for f(z) to be- analytic : The necessary and sufficient conditions for he function

$$w = f(z) = u(x,y) + iv(x,y)$$

To be analytic in a region R, are

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ are continuous functions of x and y in the region R.

(ii) $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

The conditions In (ii) are known as Cauchy-Riemann equations or briefly C-R equations.

Proof : (a) Necessary condition : Let $f(z) = u + iv$ be a single-valued function possessing partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at each point of a region R and satisfying C-R equations.

i.e., $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

We shall show that $f(z)$ is analytic i.e., $f'(z)$ exists at every point of the region R.

By Taylor's theorem for functions of two variables, we have, on omitting second and higher degree terms of δx and δy.

$$f(z+\delta z) = u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)$$

$$= \left[u(x,y) + \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y\right)\right] + i\left[v(x,y) + \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right)\right]$$

$$= \left[u(x,y) + iv(x,y)\right] + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

$$= f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial u}{\partial y}\right)\delta y$$

Or $$f(z+\delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial x}\right)\delta x + \left(-\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial x}\right)\delta y \qquad \text{[Using C-R equations]}$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)i\delta y \quad \left[\because -1 = i^2\right]$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\delta x + i\delta y) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta z \quad \left[\because \delta x + i\delta y = \delta z\right]$$

$\Rightarrow$ $$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Thus $f'(z)$ exists, because $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$ exist.

Hence $f(z)$ is analytic.

Q. 3. (b) If $f(z) = u + iv$ is analytic function, find $f(z)$,k if $u - v = e^x(\cos y - \sin y)$.

Ans. $$u = v = e^x(\cos y - \sin y)$$

$\therefore$ $$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = e^x(\cos y - \sin y) \qquad \text{...(i)}$$

& $$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = e^x[-\sin y - \cos y]$$

$$-\frac{\partial u}{\partial x}-\frac{\partial u}{\partial x}=e^x[-\sin y-\cos y] \qquad \text{...(ii)}$$

Subtracting (ii) from (i)

$$\frac{\partial u}{\partial x}=e^x\cos y-e^x\sin y+e^x\sin y+e^x\cos y$$

$$=2e^x\cos y$$

Adding (i) & (ii)

$$-\frac{2dv}{dx}=+2e^x\sin y$$

Thus, $$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial u}{\partial x}$$

and putting $x=z$ & $y=0$

$$f'(z)=\left(e^z\cos 0+ie^z\sin 0\right)$$

$$=e^z$$

$$f(z)=\int e^z dz+c$$

$$=e^z+c$$

Q. 4. (a) Show that $\int_c (z+1)dz=0$ where C is the boundary of the square whose vertices are at the points $z=0, z=1, z=1+i$ and $z=i$.

Ans. $\int_c (z+1)dz$

$$=\int_0^1 (z+1)dz+\int_1^{1+i}(z+1)dz+\int_{i+1}^{i}(z+1)dz+\int_i^0 (z+1)dz$$

$$=\left(\frac{z^2}{2}+z\right)_0^1+\left(\frac{z^2}{2}+z\right)_1^{1+i}+\left(\frac{z^2}{2}+z\right)_{i+1}^{i}+\left(\frac{z^2}{2}+z\right)_i^0$$

$$=\left(\frac{3}{2}-0\right)+\left[\left\{\frac{(1+i)^2}{2}+(1+i)\right\}-\frac{3}{2}\right]+\left[\left(\frac{i^2}{2}+i\right)-\left\{\frac{(i+1)^2}{2}+(i+1)\right\}\right]$$

$$+\left[0-\left(\frac{i^2}{2}+1\right)\right]$$

$$= 0 = \text{RHS}.$$

Q. 4. (b) Evaluate $\int_c \frac{z^3+z+1}{z^2-7z+2}dz$**, where C is the ellipse**

$$4x^2+9y^2=1.$$

Ans. We know $\quad z = x+iy \Rightarrow dz = dx+ixy$

First we have to check whether the function is analytic considering,

$$z^3+z+1=(x+iy)^3+(x+iy)+1$$

$$=\left(x^3-3xy^2+x+1\right)+i\left(3x^2y-y^3+y\right)$$

Now, $\quad \frac{\partial u}{\partial x}=3x^2-3y^2+1;\ \frac{\partial u}{\partial y}=-6xy;\ \frac{\partial v}{\partial x}=6xy;\ \frac{\partial v}{\partial y}=3x^2-3y^2+$

i.e. Cauchy Riemann equations are satisfied, hence the above function is analytical.

Considering $\quad z^2-7z+2=(x+iy)^2-7(x+iy)+2$

$$=\left(x^2-y^2-7x+2\right)+i(2xy-7y)$$

$$\frac{\partial u}{\partial x}=2x-7;\ \frac{\partial v_1}{\partial x}=2y;\ \frac{\partial u}{\partial y}=-2y;\ \frac{\partial v}{\partial y}=2x-7$$

Hence Cauchy's Riemann's equations are satisfied the function is analytic.

Therefore, the function $\frac{z^3+z+1}{z^2-7z+2}$ is analytic everywhere with in and on $4x^2+9y^2=1$.

$\therefore$ By Cauchy's integral theorem.

$$\oint_c f(z)dz=0.$$

Q. 4. (c) Expand $\dfrac{e^z}{(z-1)^2}$ about $z = 1$.

Ans. Let $\quad z-1=t$ or $z=t+1$

$$\therefore \qquad f(z)=\frac{e^{(t+1)}}{t^2}=\frac{e}{t^2}.e^t$$

$$=\frac{e}{t^2}\left[1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\ldots\ldots\right]$$

$$=e\left[\frac{1}{t^2}+\frac{1}{t}+\frac{1}{2!}+\frac{t}{3!}+\frac{t^2}{4!}+\ldots\ldots\right]$$

$$=e\left[\frac{1}{(z-1)^2}+\frac{1}{(z-1)}+\frac{1}{2!}+\frac{(z-1)}{3!}+\frac{(z-1)^2}{4!}+\ldots\ldots\right]$$

$$=e\left[(z-1)^{-2}+(z-1)^{-1}+\frac{1}{2!}+\frac{1}{3!}(z-1)+\frac{1}{4!}(z-1)^2+\ldots\right].$$

Q. 5. (a) Evaluate

$\int_c \dfrac{e^z}{\cos \pi z}dz$, where C is the unit circle $|z|=1$. How many poles the function $\dfrac{e^z}{\cos \pi z}$.

Ans. $\qquad f(z)=\dfrac{e^z}{\cos \pi z}$ has simple poles at

$$z=\pm\frac{1}{2},\ \pm\frac{3}{2},\ \pm\frac{5}{2},\ldots\ldots$$

Of which only $z=\pm\frac{1}{2}$ lies inside the circle $|z|=1$.

Residue of $f(z)$ at $z=\frac{1}{2}$ is

$$\lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)f(z) = \lim_{z\to\frac{1}{2}}\frac{\left(z-\frac{2}{1}\right)e^{2}}{\cos\pi z} \qquad \left|\text{Form } \frac{0}{0}\right|$$

$$= \lim_{z\to\frac{1}{2}}\frac{\left(z-\frac{1}{2}\right)e^{z}+e^{z}}{-\pi\sin\pi z} \qquad \text{(By L Hospital rule)}$$

$$= \frac{e^{1/2}}{-\pi}$$

Similarly, residue of $f(z)$ at $z=-\frac{1}{2}$ is, $e^{\frac{-1/2}{\pi}}$.

$\therefore$ By residue theorem,

$$\int_c \frac{e^{z}}{\cos\pi z}dz = 2\pi i \text{ (sum of residues)}$$

$$= 2\pi i\left(-\frac{e^{1/2}}{\pi}+\frac{e^{-1/2}}{\pi}\right) = -4i\left(\frac{e^{1/2}-e^{-i/2}}{2}\right)$$

$$= -4i\sinh\frac{1}{2}.$$

Q. 5. (b) Evaluate $\int_{-\infty}^{\infty}\frac{x^2}{x^6+1}dx$ by residues.

Ans. Now let,

$$x^3 = X$$

$$3x^2dx = dx$$

$$x^2dx = \frac{dX}{3}$$

$$\therefore \quad \frac{1}{3}\int_{-\infty}^{\infty}\frac{dX}{X^2+1}.$$

The poles of the $\phi(z)=\frac{1}{X^2+1}$ are obtained by solving

$$z^2+1=0$$

$$z=(-1)^{1/2}=(\cos\pi+i\sin\pi)^{1/2}$$

$$=[\cos(2x\pi+\pi)+i\sin(2n\pi+\pi)]^{1/2}$$

$$z=\left[\cos\left(\frac{2n+1}{2}\right)\pi+i\sin\left(\frac{2n+1}{2}\right)\pi\right]$$ By De Moivre's theorem

Where n = 0, 1, 2

$n=0,$ $\qquad z=\cos\pi/2+i\sin\frac{\pi}{2}=i$

$n=1,$ $\qquad z=\cos3\pi/2+i\sin\frac{3\pi}{2}=-1$

$n=2,$ $\qquad z=\cos\frac{5\pi}{2}+i\sin\frac{5\pi}{2}=i$

Reduce of $\phi(z)$ at $_{z}e^{i\pi/2}$ is

$$\lim_{z\to e^{i\pi/2}}\frac{(z-e^{i\pi/2})}{z^2+1}=\lim_{z\to e^{i\pi/2}}\cdot\frac{1}{2z}=\frac{1}{ze^{i\pi/2}}$$

$$=\frac{1}{2}e^{-i\pi/2}$$

Residue at $z=e^{3i\pi/2}$ is $\frac{1}{2}e^{-3i\pi/2}$

Residue at $z=e^{5i\pi/2}$ is $\frac{1}{2}e^{-5i\pi/2}$

$$\therefore \qquad \int_0^\infty\frac{x^2}{x^6+1}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2}{x^6+1}$$

$$= \frac{1}{2} 2\pi i \text{ [Sum of residues in the upper half of s plane]}$$

$$= \frac{1}{2} 2\pi i . \frac{1}{3}\left[\frac{1}{2}(-i + i = i)\right]$$

$$= \frac{\pi}{3} i\left(-\frac{1}{2} i\right) = \frac{-\pi}{6} i^2$$

$$= \pi / 6 .$$

Part-C

Q. 6. (a) In a bolt factory, machines A, B and C manufacture 25%, 35% and 40$ of the total output. Of their output 5%, 4% and 2% are defective bolts. A bolt is drawn at random from the product and is found to be defective. What are the probabilities that it was manufactured by the machines A, B or C?

Ans. Let E_1, E_2 & E_3 denote the events that a bolt is selected from machine A, B & C respectively.

$$P(E_1) = \frac{25}{100}, \; P(E_2) = \frac{35}{100}, \; P(E_3) = \frac{40}{100}$$

$$P(D/E_1) = \frac{5}{100}, \; P(D/E_2) = \frac{4}{100}, \; P(D/E_3) = \frac{2}{100}$$

By Baye's theorem,

$$P(D/E_1) = \frac{P(E_1)P(D/E_1)}{\sum_{i=1}^{3} P(E_i)P(D/E_i)} = \frac{\frac{25}{100} \times \frac{5}{100}}{\frac{25}{100} \times \frac{5}{100} + \frac{35}{100} \times \frac{4}{100} + \frac{40}{100} \times \frac{2}{(100)^2}}$$

$$= \frac{125}{125 + 140 + 80} = \frac{125}{345} = \frac{25}{69}$$

$$P\left(\frac{E_2}{D}\right) = \frac{P(E_2)P(D/E_2)}{\sum_{i=1}^{3} P(E_i)P(D/E_i)} = \frac{\frac{35}{100} \times \frac{4}{100}}{\frac{125}{(100)^2} + \frac{140}{(100)^2} + \frac{80}{(100)^2}}$$

$$P\left(\frac{E_3}{D}\right)=\frac{P(E_3)P(D/E_3)}{\sum_{i=1}^{3}P(E_i)P(D/E_i)}=\frac{\frac{40}{100}\times\frac{2}{100}}{\frac{25}{(100)^2}\times\frac{5}{(100)^2}+\frac{35}{(100)^2}\times\frac{4}{100}+\frac{40}{100}\times\frac{2}{100}}$$

$$=\frac{80}{125+140+80}=\frac{80}{345}+\frac{16}{69}.$$

Q. 6. (b) A function is defined as under :

$$f(x)=\frac{1}{k},\ x_1 \le x \le x_2$$

$$=0 \qquad \text{otherwise.}$$

Find the cumulative distribution of the variable x when k satisfies the requirement for f(x) to be a density function.

Ans.

$$f(x)=\frac{1}{k},\ x_1 \le x \le x_2$$

$$= 0, \text{ else where}$$

$\because$ $f(x)$ is an density function.

$\therefore$
$$\int_{-\infty}^{\infty} f(x)dx = 1 \quad \int_{x_1}^{x_2}\frac{1}{k}dx = 1$$

$\Rightarrow$
$$k = x_2 - x_1$$

$\therefore$ Cumulative distribution function,

$$F(x) = 0 \text{ if } x < x_1$$

$$= \int_{x_1}^{x}\frac{1}{k}dx$$

$$= \frac{x-x_1}{x_2-x_1}; \text{ if } x_1 \le x \le x_2$$

$$= 1 \text{ if } x \ge x_2.$$

Q. 7. (a) In a lot of 500 solenoids 25 are defective, find the probability of 0, 1, 2, 3, defective solenoids in a random sample of 20 solenoids.

Ans. The probability of solenoid to be defective

$$=\frac{25}{500}=\frac{1}{20}$$

The probability of 0 defective

$$={}^{20}C_0\left(\frac{1}{20}\right)^0\left(\frac{19}{20}\right)^{20}=0.3585$$

The probability of 1 defective

$$={}^{20}C_1\left(\frac{1}{20}\right)^1\left(\frac{19}{20}\right)^{19}=0.3774$$

The probability of 2 defective

$$={}^{20}C_2\left(\frac{1}{20}\right)^2\left(\frac{19}{20}\right)^{18}=0.1887$$

The probability of 3 defective

$$={}^{20}C_3\left(\frac{1}{20}\right)^3\left(\frac{19}{20}\right)^{17}=0.0596$$

The probability of r^{th} defective

$$={}^{20}C_8\left(\frac{19}{20}\right)^{20-r}\left(\frac{1}{20}\right)^r$$

where r = 0, 1, 2, 3.

Q. 7. (b) Fit a Poisson distribution to the set of observations :

x :	0	1	2	3	4
f :	122	60	15	2	1

Ans. $$\text{Mean}=\frac{\Sigma f_i x_i}{\Sigma f_i}=\frac{60+36+6+4}{200}=0.5$$

$\therefore$ Mean of Poisson distribution i.e., = 0.5

Hence the theoretical frequency for r success is

$$\frac{Ne^{-m}(m)^r}{r!} = \frac{200e^{-0.5}(0.5)^r}{r!} \text{ where } r = 0, 1, 2, 3, 4$$

∴ The theoretical frequencies are,

x :	0	1	2	3	4	
f :	121	61	15	2	0	$(\because e^{-5} = 0.61)$

Q. 8. (a) Using Simplex method :

Maximize $Z = 3x_1 - 2x_2 + 4x_3$

subject to :

$$x_1 + 2x_2 + x_3 \le 8$$

$$2x_1 - x_2 + x_3 \ge 2$$

$$4x_1 - 2x_2 - 3x_3 = -6;$$

$$x_1, x_2 \ge 0.$$

Ans. Max. $z = 3x_1 + 2x_2 + 4x_3$

Subject to $x_1 + 2x_2 + x_3 \le 8$; $2x_1 - x_2 + x_3 \ge z$

$$4x_1 - 2x_2 - 3x_3 = -6; \; x_1, x_2 \ge 0$$

Step : The first constraint involves $\le$. We introduce only a slack variable s_1 thereby getting

$$x_1 + 2x_2 + x_3 + s_1 = 8$$

The 2nd constraint is greater than type. We introduce a surplus variable s_2 and artificial variable A_1 thereby getting

$$2x_1 - x_2 + x_3 - s_2 + A_1 = 2$$

The 3rd constrain is strict equality and requires neither a stack variable or surplus variable. We add only an artificial variable A_2 these by getting and multiplying by (–1) we get

$$-4x_1 + 2x_2 - 3x_3 + A_2 = 6$$

∴ The standard form of LPP problem becomes,

$$\text{Max.,} \quad Z = 3x_1 - 2x_2 + 4x_3 + 0s_1 + 0s_2 + MA_1 + MA_2$$

$$x_1 + 2x_2 + x_3 + s_1 = 8$$

$$2x_1 - x_2 + x_3 - s_2 + A_1 = 2$$

$$-4x_1 + 2x_2 - 3x_3 + A_2 = 6$$

Where $x_1, x_2, x_3, s_1, s_2, A_1, A_2 \geq 0$ and M is a large positive number.

Step 2 : Since we have 3 equations in 7 variable, a solution is obtained by setting $7 - 3 = 4$ variables equal to zero and solve for remaining variables.

Now an initial basic feasible solution can be obtained by $x_1 = x_2 = x_3 = s_2 = 0$. Therefore the initial basic feasible solution is,

$$s_1 = 8,\ A_1 = 2,\ A_2 = 6$$

and $$Z = 2M + 6M = 8M$$

Therefore the initial basic solution is tabulated as :

Simplex table I :

C_B	Basic	C_j	3	-2	4	0	0	M	N	Ratio
		Solution	x_1	x_2	x_3	s_1	s_2	A_1	A_2	x_B / x_1
	$b(= x_B)$									
0	s_1	8	1	2	1	1	0	0	0	$\frac{8}{1} = 8$
M	A_1	2	2	-1	1	0	-1	1	0	$\frac{2}{2} = 1$
M	A_2	6	-4	2	-3	0	0	0	1	$-\frac{6}{4} = -\frac{2}{3} \rightarrow$
$Z = 8M$		Z_j	$-2M$	M	$-2M$	0	$-M$	M	M	
		$c_j = c_j - z_j$	$3-2M$	$-2-M$	$-2-M \uparrow$	$4+2M$	0	M	0	0

Some entries in the C_j row being –ve the current solution is not optimal.

Step 3 : Largest negative entry in given is $-2-M$ which lies in x_2 column. Therefore the incoming variable is x_2. The ratio –2/3 is minimum in A_2 row the outgoing basic variable is A_2. Key element is 2. (In further simplex table we will not compute A_2 column) Now key row is

2 x_1 3 –2 1 –3/2 0 0 0

Transformation of R_1 key column entry in R_1 is 2

$$\therefore \quad R_1(new) = R_1(old) - 2R_3(new)$$

$$8-2(3)=2$$

$$1-2(-2)=5$$

$$2-2(1)=0$$

$$1-2\left(-\frac{3}{2}\right)=4$$

$$1-2(0)=1$$

$$0-2(0)=0$$

$$0-2(0)=0$$

Transformation of R_2 key column entry in R_2 is (–1)

$$R_2(new) = R_2(old) + R_3(new)$$

$$=2+3=5$$

$$2-2=0$$

$$-1+1=0$$

$$1-3/2=-1/2$$

$$0+0=0$$

$$-1+0=0$$

$$1+0=0$$

New simplex table is given as,

Simplex Table II

C_B	Basis	$C_j \rightarrow$	3	–2	4	0	0	M	Ratio
		solution	x_1	x_2	x_3	s_1	s_2	A_1	x_B/x_3

0	s_1	2	5	0	4	1	0	0	1/2
M	A_1	5	0	0	$-\frac{1}{2}$	0	0	0	$-10 \rightarrow$
2	x_2	3	-2	1	$-\frac{3}{2}$	0	0	0	-2
$Z = 6+5M$			Z_j	-4	2	$-3-\frac{M}{2}$	0	0	0
					$\uparrow$				

Some entries in C_j row being negative the current solution is not optimal.

Step 4 : Largest negative entry in C_j row $-3-M/2$ which lies in x_3-column. Therefore the incoming variable is x_3. The ratio (–10) is minimum in the A_1 row, therefore the outgoing variable is A_1. Key constant is $-\frac{1}{2}$. New key row,

4	x_3	-16	0	0	1	0	0	0

Transformation of R_1. Key column entry in R_1 is 4

$$\therefore \quad R_{1(new)} = R_{1(old)} - 4R_{2(new)}$$

$$2-4(-16)=66$$

$$5-4(0)=5$$

$$0-4(0)=0$$

$$4-4(1)=0$$

$$1-4(0)=1$$

$$0-4(0)=0$$

$$0-4(0)=0$$

Transformation of R_3. Key column entry in R_3 is $\left(-\frac{3}{2}\right)$

$\therefore$ $$R_{3(new)} = R_{3(old)} + \frac{3}{2}R_{2(new)}$$

$$3+\frac{3}{2}(-16) = -21$$

$$-2+\frac{3}{2}(0) = -2$$

$$1+\frac{3}{2}(0) = 1$$

$$-\frac{3}{2}+\frac{3}{2}(1) = 0$$

$$0+\frac{3}{2}(0) = 0$$

$$0+\frac{3}{2}(0) = 0$$

The new simplex table is given as,

Simplex table III :

C_B	Basic	C_j	3	−2	4	0	0	M	N	Ratio
		Solution	x_1	x_2	x_3	s_1	s_2	A_1	A_2	x_B / x_2
	$b(=x_B)$									
0	s_1	66	5	0	0	1	0	0		
4	x_3	−16	0	0	1	0	0	0		
2	x_2	−21	−2	1	0	0	0	0		
$z = -106$			z_j	−4	2	4	0	0	0	
		$c_j = c_j - z_j$	−1	−4	0	0	0	0		
				↑						

Contining in this way we find the optimal solution.