Seat No.: _____ Enrolment No._______

GUJARAT TECHNOLOGICAL UNIVERSITY

B.E. Sem-II [All Branch] examination June 2009

Subject code: 110009
Subject Name: Maths - II

Date: 17/06/2009 **Time: 10:30am-1:30pm**

Total Marks: 70

Instructions:

1. **Attempt all questions.**
2. **Make suitable assumptions wherever necessary.**
3. **Figures to the right indicate full marks.**

Q.1 **ATTEMPT THE FOLLOWING:**

(a) Define rank of the matrix. Find the rank of the matrix **03**

$$A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix}$$

(b) Let R^3 have the Euclidean inner product. Use the Gram-Schmidt process to transform the basis vectors $u_1 = (1, 1, 1)$, $u_2 = (-1, 1, 0)$ and $u_3 = (1, 2, 1)$ into an orthonormal basis $\{v_1, v_2, v_3\}$. **04**

(c) Find the eigenvalues and bases for the eigenspaces of A^{25} and $A+2I$, where **03**

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

(d) i. Show that the functions $f(x) = x$ and $g(x) = \sin x$ form a linearly independent set of vectors in $C^1(-\infty, \infty)$. **01**

ii. Show that if $0 < \theta < \pi$, then $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ has no real eigenvalues and consequently no eigenvectors. **01**

iii. Define singular matrix. Find the inverse of the matrix A if it is invertible **02**

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Q.2

(a) Determine the dimension and basis for the solution space of the system $3x_1 + x_2 + x_3 + x_4 = 0,\ 5x_1 - x_2 + x_3 - x_4 = 0$ **03**

(b) Justify your answer. Why the following sets are not vector space under the given operations? **02**

i. The set of all pairs of real numbers (x, y) with the operation $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $\alpha(x, y) = (2\alpha x, 2\alpha y)$.

ii. In R^3, the operations defined as under $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (z_1 + z_2, y_1 + y_2, x_1 + x_2)$

(c) Let λ be an eigenvalue of a matrix A. **02**
Then prove that (i) $\lambda + k$ is an eigenvalue of $A + kI$ (ii) $k\lambda$ is an eigenvalue of kA.

(d) i. Solve the following system by Gauss-Elimination method **04**

$2x+2y+2z=0,\ -2x+5y+2z=1,\ 8x+y+4z=-1$

ii. By using Gauss-Jordan elimination, Find the inverse of the given matrix **03**

$$A=\begin{bmatrix}\frac{1}{5} & \frac{1}{5} & \frac{-2}{5}\\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10}\\ \frac{1}{5} & \frac{-4}{5} & \frac{1}{10}\end{bmatrix}$$

OR

(d) I For which values of K and λ the following system have (i) no solution (ii)unique solution (iii) an infinite no. of solutions **04**

$x+y+z=6, x+2y+3z=10, x+2y+Kz=\lambda$

II Find basis for the row and column spaces of $A=\begin{bmatrix}1 & 4 & 5 & 4\\ 2 & 9 & 8 & 2\\ 2 & 9 & 9 & 7\\ -1 & -4 & -5 & -4\end{bmatrix}$.

Q.3

(a) Prove that $(M_n(R),+,\bullet)$ is a vector space over $\mathbf{R}$. **05**

(b) Determine whether the following spans the vector space $\mathbf{R}^3$; **03**

(i) $v_1=(2,-1,3), v_2=(4,1,2)$ and $v_3=(8,-1,8)$,

(ii) $v_1=(2,2,2),\ v_2=(0,0,3)$ and $v_3=(0,1,1)$.

(c) Let M_{22} have the inner product $\langle A, B\rangle = tr(A^T B)$. Find the cosine of the angle between A and B, where $A=\begin{bmatrix}2 & 6\\ 1 & -3\end{bmatrix}, B=\begin{bmatrix}3 & 2\\ 1 & 0\end{bmatrix}$ **02**

(d) Show that $v=\{(x,y)/x=3y\}$ is a subspace of R^2. State all possible subspaces of R^2. **02**

(e) Find the rank and nullity of the matrix $A=\begin{bmatrix}2 & 0 & -1\\ 4 & 0 & -2\\ 0 & 0 & 0\end{bmatrix}$ **02**

OR

Q.3 (a) Let $V=\{(x,y)/x,y\in\mathbf{R}, y>0\}$. Let $(a,b),(c,d)\in V$ and $\alpha\in\mathbf{R}$. Define $(a,b)+(c,d)=(a+c,b\cdot d)$ and $\alpha\cdot(a,b)=(\alpha a,b^{\alpha})$. **05**

(b) Define basis of a vector space. Let $v_1=(1,0,0), v_2=(2,2,0)$ and $v_3=(3,3,3)$. Show that the set $S=\{v_1,v_2,v_3\}$ is a basis for $\mathbf{R}^3$. **03**

(c) Define inner product space. Let $u=\langle u_1,u_2\rangle, v=\langle v_1,v_2\rangle\in\mathbf{R}^2$. Define $\langle u, v\rangle = 4u_1v_1+u_2v_1+4u_1v_2+4u_2v_2$. Prove that $(\mathbf{R}^2,\langle\cdot,\cdot\rangle)$ is an inner product space. **05**

(d) Let V be a vector space. For a nonempty set A, prove that $A\subset span(A)$. **01**

Q.4 (a) Check whether the following transformations are linear or not? **02**

(i) $T:V\to\mathbf{R}$, where V is an inner product space, and $T(u)=\|u\|$.

(ii) $T:M_{mn}\to M_{nm}$, where $T(A)=A^T$.

(b) Let $T:\mathbf{R}^2 \to \mathbf{R}^2$, defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)=\begin{bmatrix} x_1-2x_2 \\ -x_2 \end{bmatrix}$ and let $B = \{e_1, e_2\}$ and $B' = \left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \end{bmatrix}\right\}$. Then using $[T]_{B'} = P^{-1}[T]_B P$, find $[T]_{B'}$, where P is the transition matrix from B' to B. **03**

(c) Consider the basis $S=\{v_1, v_2\}$ of R^2, where $v_1=(-2,1)$ and $v_2=(1,3)$ and let $T:\mathbf{R}^2 \to \mathbf{R}^3$ be the linear transformation such that $T(v_1)=(-1,2,0)$ and $T(v_2)=(0,-3,5)$. Find a formula for $T(x_1, x_2)$, and use that to find $T(2,-3)$. **03**

(d) Define kernel of T. Let $T:\mathbf{R}^3 \to P_1$ be a linear transformation defined by $T(a_1,a_2,a_3)=(-a_1+2a_2+a_3)+(-a_2+a_3)x$. Find which of the following vectors are in $ker(T)$; $(i)\, u=(6,2,2)$, $(ii)\, u=(2,-1,1)$ and $(iii)\, u=(0,0,0)$. **02**

(e) If u and v are orthogonal unit vectors, then what is the distance between u and v? Justify your answer. **02**

(f) Define: Algebraic multiplicity of an eigenvalue. Determine the algebraic and geometric multiplicity of $A=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ if the eigenvalues of A are $\lambda=2,-1,-1$ and corresponding eigenvectors for $\lambda=2$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and for $\lambda=-1$ are $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$. **02**

OR

Q.4 (a) State Cauchy-Schwarz inequality. Verify Cauchy-Schwarz inequality for the vectors $u=(-3,1,0)$, $v=(2,-1,3)$. **02**

(b) Prove that $\langle u, v \rangle = \frac{1}{4}\| u+v \|^2 - \frac{1}{4}\| u-v \|^2$ **01**

(c) Let $T: R^4 \to R^3$ be the linear transformation given by the formula $T(x_1, x_2, x_3, x_4)=(4x_1+x_2-2x_3-3x_4, 2x_1+x_2+x_3-4x_4, 6x_1-9x_3+9x_4)$. Find a basis and dimension of $\ker(T)$, $rank(T)$ and verify the dimension theorem. **05**

(d) Find the projection of $u=(1,-2,3)$ along $v=(1,2,1)$ in $\mathbf{R}^3$. **01**

(e) Let $S, T:\mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformations given by the formulas $T(x, y)=(x+y, x-y)$ and $S(x, y)=(2x+y, x-2y)$. (i) Show that S and T are one to one, (ii) Find formula for $T^{-1}(x, y)$, $S^{-1}(x, y)$ and $(S\circ T)^{-1}(x, y)$, (iii) Verify that $(S\circ T)^{-1} = T^{-1}\circ S^{-1}$. **05**

Q.5 (a) Find a matrix P that diagonalizes $A=\begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$, and determine $P^{-1}AP$. **06**

(b) By using Cayley-Hamilton theorem, if $A=\begin{bmatrix}2 & 1 & 1\\0 & 1 & 0\\1 & 1 & 2\end{bmatrix}$, then prove that **04**

$$A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I=\begin{bmatrix}8 & 5 & 5\\0 & 3 & 0\\5 & 5 & 8\end{bmatrix}$$

(c) Find a change of variables that reduces the quadratic form $2x_1^2+2x_2^2-2x_1x_2$ to a sum of squares and express the quadratic form in terms of the new variables. **04**

OR

Q.5 (a) Find an orthogonal matrix P that diagonalizes $A=\begin{bmatrix}2 & -1 & -1\\-1 & 2 & -1\\-1 & -1 & 2\end{bmatrix}$. **06**

(b) Find the least squares solution of the linear system $Ax = b$ given by $x_1+x_2=7, -x_1+x_2=0, -x_1+2x_2=-7$ and find the orthogonal projection of b on the column space of A. **04**

(c) Describe the conic whose equation is $5x^2-4xy+8y^2-36=0$. **04**

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