B.E.

Seventh Semester Examination, 2010-2011

Digital Signal Processing (EE-407-E)

Note : Attempt any five questions. All questions carry equal marks.

Q. 1. (a) Determine whether the following system is stable or not.

$$y(n) = y^2(n-1) + x(n)$$

Ans. Given input-output equation,

$$y(n) = y^2(n-1) + x(n)$$

As an input sequence we select the bounded signal

$$x(n) = c\delta(n)$$

Where c is a constant. We also assume that $y(-1) = 0$. Then the output sequence is,

$$y(0) = c,\ y(1) = c^2,\ y(2) = c^4, \ldots\ldots\ldots\ldots, y(n) = c^{2n}$$

Clearly, the output is unbounded when $1 < 1 < 1 < \infty$. Therefore, the system is BIBO unstable, since a bounded input sequence has resulted in an unbounded output.

Q. 1. (b) State whether the following systems are : (i) Static (ii) linear (iii) time-invariant (iv) causal.

(i) $y(n) = nx(n)$ **(ii)** $y(n) = x(n^2)$ **(iii)** $y(n) = \sum_{n=-\infty}^{n} x(K)$

Ans. (i) $y(n) = nx(n)$

The system is static, linear, time variant, causal.

(ii) $y(n) = x(n^2)$

The system is not static, linear, time-invariant, non-causal.

(iii) $y(n) = \sum_{k=-\infty}^{n} x(k)$

The system is not static, linear, time-invariant, causal.

Q. 2. (a) If $y(n)$ is the convolution of $x(n)$ & $h(n)$. Then determine the convolution of $x(n-n_1)$ and $h(n-n_2)$

Ans. Given. $y(n) = x(n) * h(n)$

Taking z-transform,

$$Y(z) = X(z)H(z)$$

Now, the z-transform of,

$$x(n-n_1) \xleftrightarrow{z} z^{-n_1}X(z)$$

And $$h(n-n_2) \longleftrightarrow z^{-n_2}H(z)$$

$\therefore$ $$z^{-n_1}X(z).z^{-n_2}H(z)$$

$$= z^{-(n_1+n_2)}H(z)X(z)$$

$$= z^{-1(n_1+n_2)}y(z)$$

$\therefore$ Convolution of $x(n-n_1)$ and $h(n-n_2)$ is the inverse z-transform of $z^{-(n_1+n_2)}y(z)$.

$\therefore$ $$\boxed{y(n-(n_1+n_2)) = x(n-n_1) * h(n-n_2)}.$$

Q. 2. (b) Determine the DTFT of $x(n) = (a)^{|n|}$ for $-1 < a < 1$

Ans. First we observe that x(n) can be expressed as,

$$x(n) = x_1(n) + x_2(n)$$

Where, $$x_1(n) = \begin{cases} a^n, & n \ge 0 \\ 0, & n < 0 \end{cases}$$

And $$x_2(n) = \begin{cases} a^{-n}, & n < 0 \\ 0, & n \ge 0 \end{cases}$$

$\therefore$ $$X_1(w) = \sum_{n=-\infty}^{\infty} x_1(n)e^{-\lambda wn} \sum_{n=0}^{\infty} a^n e^{-\lambda wn} = \sum_{n=0}^{\infty} \left(ae^{-\lambda w}\right)^n$$

The summation is a geometric series that converges to

$$X_1(w) = \frac{1}{1-ae^{-\lambda w}}$$

Provided that,

$$\left|ae^{-\lambda w}\right| = |a|.\left|e^{-\lambda w}\right| = |a| < 1$$

Which is a condition that is satisfied in this problem.

Similarly, the Fourier transform of $x_2(n)$ is,

$$X_2(w) = \sum_{n=-\infty}^{\infty} x_2(n)e^{-\lambda wn}$$

$$= \sum_{n=-\infty}^{-1} a^{-n}e^{-\lambda wn}$$

$$= \sum_{n=-\infty}^{-1} \left(ae^{\lambda w}\right)^{-n}$$

$$= \sum_{k=1}^{\infty} \left(ae^{\lambda w}\right)^{k}$$

$$= \frac{ae^{\lambda w}}{1-ae^{\lambda w}}$$

By combining these two transforms, we obtain the Fourier transform of x(n) in the form,

$$X(w) = X_1(w) + X_2(w)$$

$$\boxed{X(w) = \frac{1-a^2}{1-2a\cos w + a^2}}$$

Q. 3. What is the effect of sampling in frequency domain, explain with suitable derivation.

Ans. Effect of Sampling in Frequency Domain : Let us consider such an aperiodic discrete-time signal x(n) with Fourier transform,

$$X(w) = \sum_{n=-\infty}^{\infty} x(n)e^{-\lambda wn} \quad \text{...(1)}$$

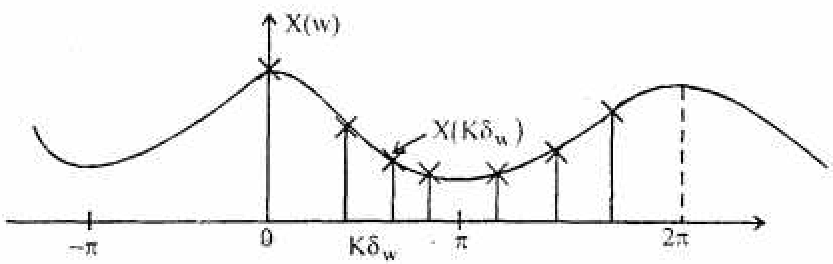


Fig. Frequency-domain sampling of the Fourier transform

Suppose that we sample X(w) periodically in frequency at a spacing of δw radians between successive samples. Since X(w) is periodic with period 2π ; only samples in the fundamental frequency range are necessary. For convenience, we take N equidistant samples in the interval $0 \le w < 2\pi$ with spacing $\delta w = 2\pi / N$, as shown in fig. First, we consider the selection of N, the number of samples in the frequency domain.

If we evaluate equation (1) at $w = 2\pi K / N$, we obtained,

$$X\left(\frac{2\pi}{N}K\right) = \sum_{n=-\infty}^{\infty} x(n)e^{-\lambda 2\pi Kn/N}, \quad K = 0, 1, \ldots\ldots, N-1 \qquad \ldots(2)$$

The summation in (2) can be subdivided into an infinite number of summations, where each sum contains N terms. Thus.

$$X\left(\frac{2\pi}{N}K\right) = \ldots\ldots\ldots\ldots + \sum_{n=-N}^{-1} x(n)e^{-\lambda 2\pi Kn/N} + \sum_{n=-N}^{N-1} x(n)e^{-\lambda 2\pi Kn/N} + \sum_{n=N}^{2N-1} x(n)e^{-\lambda 2\pi Kn/N} + \ldots\ldots\ldots$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n)e^{-\lambda 2\pi Kn/N}$$

If we change the index in the inner summation from n to n–lN and interchange the order of the summation, we obtain the result.

$$X\left(\frac{2\pi}{N}K\right) = \sum_{n=0}^{N-1}\left[\sum_{l=-\infty}^{\infty} x(n-lN)\right]e^{-\lambda 2\pi Kn/N} \qquad \ldots(3)$$

For $K = 0, 1, 2\ldots\ldots\ldots\ldots N-1$

The signal,

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN) \qquad \ldots(4)$$

Obtained by the periodic repetition of x(n) every N samples, is clearly periodic with fundamental period N. Consequently, it can be expanded in a Fourier series as :

$$x_p(n) = \sum_{K=0}^{N-1} c_K e^{\lambda 2\pi Kn/N}, \quad n = 0, 1, \ldots\ldots N-1 \qquad \ldots(5)$$

With Fourier coefficients,

$$C_K = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n)e^{-\lambda 2\pi Kn/N}, \quad K = 0, 1, \ldots\ldots\ldots\ldots N-1 \qquad \ldots(6)$$

Upon comparing equations (3) and (6), we conclude that,

$$C_K = \frac{1}{N}x\left(\frac{2\pi}{N}K\right), \quad K = 0, 1, \ldots\ldots\ldots\ldots N-1 \qquad \ldots(7)$$

Therefore, $$x_p(n) = \frac{1}{N}\sum_{K=0}^{N-1} x\left(\frac{2\pi}{N}K\right)e^{\lambda 2\pi Kn/N}, n = 0, 1, \ldots\ldots\ldots N-1 \quad \ldots(8)$$

Since, $x_p(n)$ is the periodic extension of x(n) as given by (4), it is clear that x(n) can be recovered from $x_p(n)$ if there is no aliasing in the time domain, i.e., if $x(n)$ is time-limited to less than the period N of $x_p(n)$.

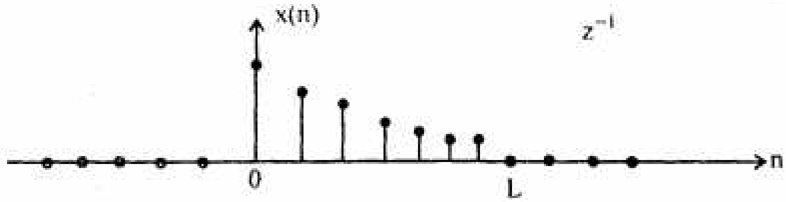


Fig. Aperiodic Sequence x(n) of length L

This situation is illustrated in fig., where without loss of generality, we consider a finite-duration sequence x(n), which is non-zero in the interval $0 \le n \le l-1$. We observe that when $N \ge L$.

$$x(n) = x_p(n),\ 0 \le n \le N-1$$

So, that x(n) can be recovered from $x_p(n)$ without ambiguity. On the other hand, if N < L, it is not possible to recover x(n) from its periodic extension due to time-domain aliasing. Thus, we conclude that the spectrum of an aperiodic discrete-time signal with finite duration L can be exactly recovered from its samples at frequencies $w_K = 2\pi K/N$, if $N \ge L$.

Q. 4. (a) Determine the z-transform of $x(n) = (-1)^n \cos\left(\frac{\pi}{3}n\right)u(n)$

Ans. Given : $$x(n) = (-1)^n \cos\left(\frac{\pi}{3}n\right)u(n)$$

$$\cos\left(\frac{\pi}{3}n\right)u(n) = \cos(w_o n)u(n),\ w_o = \frac{\pi}{3}$$

$$= \frac{1}{2}e^{\lambda w_o n}u(n) + \frac{1}{2}e^{-\lambda w_o n}u(n)$$

$\therefore$ z-transform of $\cos(w_o n)u(n)$ is,

$$= \frac{1}{2}z\left[e^{\lambda w_o n}u(n)\right] + \frac{1}{2}z\left[e^{-\lambda w_o n}u(n)\right]$$

Now, $$e^{\lambda w_o n}u(n) \xleftrightarrow{z} \frac{1}{1-e^{\lambda w_o}z^{-1}} \quad \therefore\ \text{ROC}: |z| > 1$$

And $$e^{-\lambda w_0 n}u(n) \xleftrightarrow{z} \frac{1}{1-e^{-\lambda w_0}z^{-1}} \quad \text{ROC}: |z| > 1$$

$\therefore$ z-transform of $\cos(w_0 n)u(n)$

$$\cos(w_0 n)u(n) \xleftrightarrow{z} \frac{1-z^{-1}\cos w_0}{1-2z^{-1}\cos w_0 + z^{-2}}; \qquad \text{ROC}:|z| > 1$$

$$\therefore \quad \cos\left(\frac{\pi}{3}n\right)u(n) \xleftrightarrow{z} \frac{1-z^{-1}\cos\left(\frac{\pi}{3}\right)}{1-2z^{-1}\cos(\pi/3)+z^{-2}}, \qquad \text{ROC}: |z| > 1$$

We know that

$$a^n x(n) \xleftrightarrow{z} X\left(a^{-1}z\right)$$

$$(-1)^n \cos\left(\frac{\pi}{3}n\right)u(n) \leftrightarrow \frac{1+z^{-1}\cos(\pi/3)}{1+2z^{-1}\cos(\pi/3)+z^{-2}} \qquad \text{ROC}:|z| > 1$$

Q. 4. (b) Determine the inverse z-transform of

$$X(z) = \log\left(1-2z^{-1}\right)$$

Ans. Given $$X(z) = \log\left(1-2z^{-1}\right)$$

Now, $$\frac{d}{dz}X(z) = \frac{2z^{-2}}{1-2z^{-1}}$$

If we multiply both sides of this equation by (–z), we have

$$Y(z) = -z\frac{d}{dz}X(z) = -\frac{2z^{-2}}{1-2z^{-1}}$$

Now by applying derivative property.

$$nx(n) = y(n) = -(2)^n u(n-1)$$

$$\therefore \quad \boxed{x(n) = -\frac{1}{n}(2)^n u(n-1)} \qquad \textbf{Ans.}$$

Q. 5. (a) Design a linear phase low pass FIR filter with cut-off frequency $w_0 = 1$ rad/sample, length of the filter should be 7. Use Hanxing window.

Ans. Given length of the filter M = 7, cut-off frequency $w_C = 1$ rad/sample.

The unit sample response is,

$$h_d(n) = \begin{cases} \dfrac{\sin(n-3)}{\pi(n-3)}, & n \neq 3 \\ \dfrac{1}{\pi}, & n = 3 \end{cases} \quad \text{...(1)}$$

Calculation of $h_d(n)$:

S. No.	n	Value of Coefficient $h_d(n)$ according to equation (1)
1	0	$h_d(0) = 0.01497$
2	1	$h_d(1) = 0.14472$
3	2	$h_d(2) = 0.26785$
4	3	$h_d(3) = 1/\pi = 0.31831$
5	4	$h_d(4) = 0.26785$
6	5	$h_d(5) = 0.14472$
7	6	$h_d(6) = 0.01497$

We know that manning window function is given by

$$w(n) = \frac{1}{2}\left[1 - \cos\frac{2\pi n}{M-1}\right] \quad \text{...(2)}$$

S. No.	n	$W(n) = \frac{1}{2}\left[1-\cos\frac{2\pi n}{M-1}\right]$, **M = 7**
1	0	w(0) = 0
2	1	w(1) = 1/4
3	2	w(2) = 3/4
4	3	w(3) = 1
5	4	w(4) = 3/4
6	5	w(5) = 1/4
7	6	w(6) = 0

Now h(n) is given by,

$$h(n) = h_d(n)\,w(n)$$

Q. 5. (b) Explain the design of IIR filter using bilinear transformation.

Ans. Design of IIR Filter Using Bilinear Transformation : The bilinear transformation is a conformal mapping that transforms the $j\Omega$ – axis into the unit circle in the z-plane only once, thus avoiding aliasing of frequency components. Furthermore, all points in the LMP of S are mapped inside the unit circle in the z-plane and all points in the RMP of S are mapped into corresponding points outside the unit circle in the z-plane.

The bilinear transformation can be linked to the trapezoidal formula for numerical integration. For example, let us consider an analog linear filter with system function,

$$M(s) = \frac{b}{s+a} \quad \text{...(1)}$$

This system is also characterized by the differential equation,

$$\frac{dy(t)}{dt} + ay(t) = bx(t) \quad \text{...(2)}$$

Instead of substituting a finite difference for the derivative, suppose that we integrate the derivative and approximate the integral by the trapezoidal formula. Thus,

$$y(t) = \int_{t_0}^{t} y'(\beta)d\beta + y(t_0) \quad \text{...(3)}$$

Where y'(t) denotes the derivative of y(t). The approximation of the integral in equation (3) by the trapezoidal formula at $t = nT$ and $t_0 = nT - T$ yields.

$$y(nT) = \frac{T}{2}\left[y'(nT) + y'(nT-T)\right] + u(nT-T) \quad \text{...(4)}$$

Now the differential equation in equation (4) evaluated at $t = nT$ yields.

$$y'(nT) = -ay(nT) + bx(nT) \quad \text{...(5)}$$

We use (5) to substitute for the derivative in equation (4) and thus obtain a difference equation for the equivalent discrete-time system with $y(n) \equiv y(nT)$ and $x(n) \equiv x(nT)$, we obtain the result.

$$\left(1+\frac{aT}{2}\right)y(n) - \left(1-\frac{aT}{2}\right)y(n-1) = \frac{bT}{2}\left[x(n) + x(n-1)\right] \quad \text{...(6)}$$

The z-transform of this equation is,

$$\left(1+\frac{aT}{2}\right)y(z) - \left(1-\frac{aT}{2}\right)z^{-1}y(z) = \frac{bT}{2}\left[1+z^{-1}\right]x(z)$$

$\therefore$ System function of the equivalent digital filter is,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(bT/2)\left(1+z^{-1}\right)}{1+\frac{aT}{2} - (1-aT/2)z^{-1}}$$

Or,
$$H(z)=\frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+a} \quad ...(7)$$

Clearly, the mapping from S-plane to Z-plane is,

$$S=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \quad ...(8)$$

This is called the bilinear transformation.

Q. 6. (a) Draw and explain the structure for linear phase FIR filter.

Ans. Structure for Linear Phase FIR Filter : Linear phase filters have a unit sample response that is either symmetric,

$$h(n)=h(N-m)$$

Or antisymmetric,

$$h(n)=-h(N-n)$$

This symmetry may be exploited to simply the network structure. For example, if N is even and h(n) is symmetric (type I filter).

$$y(n)=\sum_{K=0}^{N} n(k)x(n-k)=\sum_{K=0}^{\frac{N}{2}-1} n(k)\left[x(n-k)+x(n-N+k)\right]+h\left(\frac{N}{2}\right)x\left(n-\frac{N}{2}\right)$$

Therefore, forming the sums $\left[x(n-k)+x(n-N+k)\right]$ prior to multiplying by h(k) reduces the number of multiplications. The resulting structure is shown in fig. (a). If N is odd and h(n) is symmetric (type II filter), the structure is as shown in fig. 1(b). There are similar structures for the antisymmetric (type III and IV) linear phase filters.

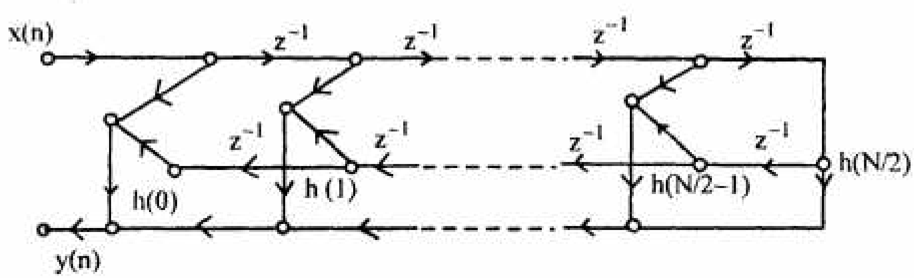


Fig. (a)

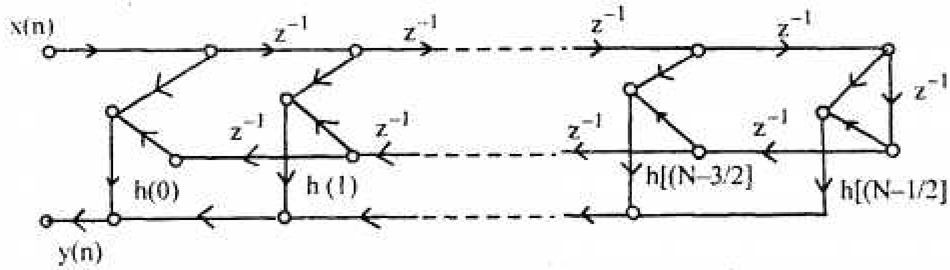


Fig. (b)

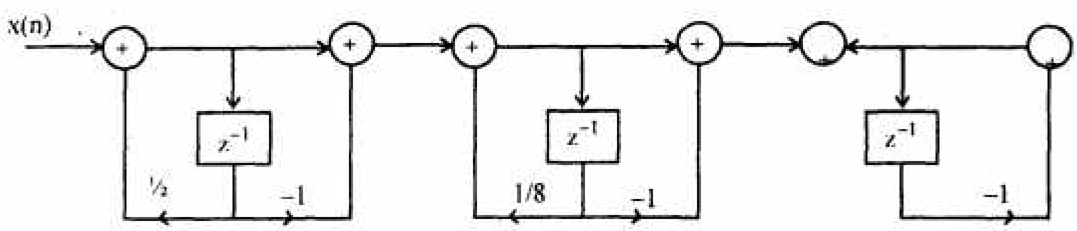
Q. 6. (b) Draw the structure of cascade & parallel realization of,

$$H(z) = \frac{\left(1-z^{-1}\right)^3}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{8}z^{-1}\right)}$$

Ans. Cascade Realization : Given transfer function H(z) can be expressed as,

$$H(z) = \frac{\left(1-z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)} \cdot \frac{\left(1-z^{-1}\right)}{\left(1-\frac{1}{8}z^{-1}\right)} \cdot \left(1-z^{-1}\right)$$

$$= H_1(z).H_2(z).H_3(z)$$



Parallel Realization : H(z) can be expressed as,

$$H(z) = \frac{(z-1)^3}{z\left(z-\frac{1}{2}\right)\left(z-\frac{1}{8}\right)}$$

To determine the parallel realization, the partial fraction of H(z)/z can be found as,

$$F(z) = H(z)/2 = \frac{(z-1)^3}{z^2\left(z-\frac{1}{2}\right)\left(z-\frac{1}{8}\right)}$$

$$= \frac{A_1}{z^2} + \frac{A_2}{z} + \frac{A_3}{\left(z-\frac{1}{2}\right)} + \frac{A_4}{\left(z-\frac{1}{8}\right)}$$

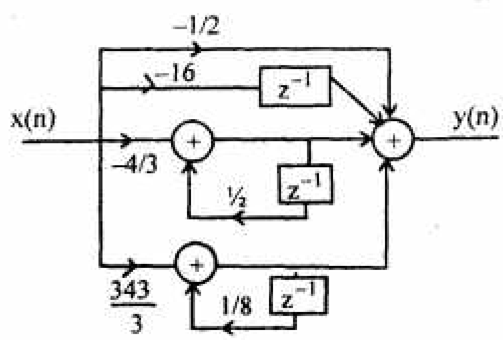
$\therefore$
$$A_1 = \left.\frac{(z-1)^3}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{8}\right)}\right|_{z=0} = -16$$

$$A_2 = \frac{d}{dz}\left[\frac{(z-1)^3}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{8}\right)}\right]_{z=0} = -112$$

$$A_3 = \left.\frac{(z-1)^3}{z^2\left(z-\frac{1}{8}\right)}\right|_{z=1/2} = -4/3$$

$$A_4 = \left.\frac{(z-1)^3}{z^2\left(z-\frac{1}{2}\right)}\right|_{z=1/8} = \frac{343}{3}$$

$\therefore$
$$H(z) = -112 - 16z^{-1} - \frac{4}{3}\frac{1}{\left(1-\frac{1}{2}z^{-1}\right)} + \frac{343}{3}\frac{1}{\left(1-\frac{1}{8}z^{-1}\right)}$$



Q. 7. Explain the need & use of multirate DSP. Also discuss the decimation process in detail.

Ans. Need and Use of Multirate DSP : In many applications of DSP, one is forced with the problem of changing the sampling rate of a signal, either increasing it or decreasing it by some amount. For example, in telecommunication systems that transmit and receive different type of signals, there is a requirement to process the various signals at different rates commensurate with the corresponding bandwidths of the signals. The process of converting a signal from a given rate to a different rate is called sampling rate conversion. In turn, systems that employ multiple sampling rates in the processing of digital signals are called multirate digital signal processing systems.

The applications that employ multirate DSP are implementation of narrowband filters, phase shifters, filter banks, subband speech coders, quadrature mirror filters and transmultiplexers.

Decimation Process : Suppose that we would like to reduce the sampling rate by an integer faction M. With a new sampling period $T_s^{'} = MT_s$, the resampled signal is,

$$x_d(n) = x_a(nT_s^{'}) = x_a(nMT_s) = x(nM)$$

Therefore, reducing the sampling rate by an integer factor M may be accomplished by taking every Mth sample of x(n). The system for performing this operation, called a down-sampler, is shown in fig. (a). Down-sampling generally results in aliasing. Specifically, recall that the DTFT of $x(n) = x_a(nT_s)$ is

$$X(e^{\lambda w}) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a\left(\lambda\frac{w}{T_s} - \lambda\frac{2\pi k}{T_s}\right)$$

Similarly, the DTFT of $x_d(n) = x(nM) = x_a(nMT_s)$ is

$$X_d(e^{\lambda w}) = \frac{1}{MT_s}\sum_{r=-\infty}^{\infty} X_a\left(\lambda\frac{w}{MT_s} - \lambda\frac{2\pi r}{MT_s}\right)$$

Note that the summation index r in the expression for $X_d(e^{\lambda\omega})$ may be expressed as,

$$r = i + KM$$

Where, $-\infty < k < \infty$ and $0 \le i \le M-1$

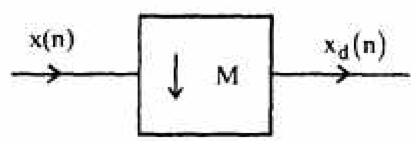


Fig. (a)

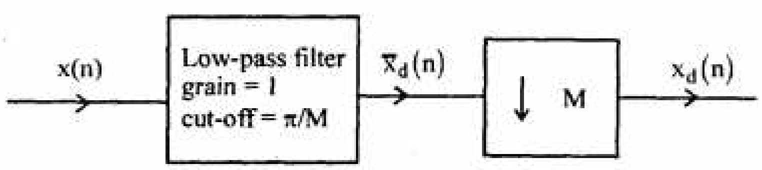


Fig. (b)

Therefore, $X_d\left(e^{\lambda w}\right)$ may be expressed as,

$$X_d\left(e^{\lambda w}\right)=\frac{1}{M}\sum_{i=0}^{M-1}\left[\frac{1}{T_s}\sum_{k=-\infty}^{\infty}X_a\left(\lambda\frac{w}{MT_s}-\lambda\frac{2\pi k}{T_s}-\lambda\frac{2\pi i}{MT_s}\right)\right]$$

The term inside the square brackets is,

$$X\left(e^{\lambda(w-2\pi i)}/M\right)=\frac{1}{T_s}\sum_{k=-\infty}^{\infty}X_a\left(\lambda\frac{(w-2\pi i)}{MT_s}-\lambda\frac{2\pi k}{T_s}\right)$$

Thus, the relationship between $X\left(e^{\lambda w}\right)$ and $X_d\left(e^{\lambda w}\right)$ is,

$$X_d\left(e^{\lambda w}\right)=\frac{1}{M}X\left(e^{\lambda(w-2\pi k)/M}\right)$$

Therefore, in order to prevent aliasing, x(n) should be filtered prior to down-sampling with a low-pass filter that has a cut-off frequency $w_c=\frac{\pi}{M}$. The cascade of a low-pass filter with a down-sampler illustrated in fig. (b) is called a decimeter.

Q. 8. Write the short notes on the following :

(a) Properties of ROC

(b) One sided Z-transform & its time-shifting properties.

Ans. (a) Properties of ROC : The ROC is, in general, an annulus of the form,

$$\alpha<|z|<\beta$$

If $\alpha=0$, the ROC may also include the point $z=0$, and if $\beta=\infty$, the ROC may also include infinity. For a rational X(z), the ROC will contain no poles. Listed below are three properties of the ROC :

1. A finite-length sequence has a Z-transform with a ROC that includes the entire z-plane except, possibly, $z=0$ and $z=\infty$. The point $z=\infty$ will be included if $x(n)=0$ for $n<0$, and the point $z=0$ will be included if $x(n)=0$ for $n>0$.

2. A right-sided sequence has a Z-transform with a ROC that is the exterior of a circle :

$$\text{ROC}:|z|>\alpha$$

3. A left-sided sequence has a Z-transform with a ROC that is the interior of a circle.

$$\text{ROC}:|z|<\beta.$$

(b) One sided Z-transform and its time-shifting properties :

One Sided Z-Transform : The one-sided, or unilateral, z-transform is defined by,

/

$$X_1(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

The primary use of the one-sided z-transform is to solve linear constant coefficient difference equations that have initial conditions. Most of the properties of the one-sided z-transform are the same as those for the two-sided Z-transform.

Time-Shifting Property : If x(n) has a one-sided z-transform $X_1(z)$, the one-sided z-transform of $x(n-1)$ is,

$$x(n-1) \overset{z}{\longleftrightarrow} z^{-1} X_1(z) + x(-1)$$

It is this property that makes the one-sided z-transform useful for solving difference equations with initial conditions.