B.E.

Third Semester Examination, 2009-10

MATHEMATICS-III

Q. 1. Attempt any five questions, selecting at least one question from each Part. Each question carries equal marks.

Part-A

Q. 1. (a) Find the Fourier series of $f(x) = \begin{cases} 0 & , \quad -\pi \le x \le 0 \\ x^2 & , \quad 0 \le x \le \pi \end{cases}$

which is assumed to be periodic with period 2π.

(b) Find the Fourier sine and cosine series of

$$f(x) = \begin{cases} x & , \quad 0 < x < \pi/2 \\ 0 & , \quad \pi/2 < x < \pi \end{cases}.$$

Ans.

$$f(x) = \begin{cases} 0 & , \quad -\pi \le x \le 0 \\ x^2 & , \quad 0 \le x \le \pi \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_0^{\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\left[0 + \int_0^{\pi} x^2\,dx\right]$$

$$= \frac{1}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = \frac{\pi^3}{3\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} x^2 \cos nx\,dx\right.$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[0 + 2\pi\frac{(-1)^4}{n^2} - 0\right] = \frac{2(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} x^2 \sin nx\,dx\right.$$

$$= \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\pi^2\frac{(-1)^n}{n} + \frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right]$$

$$= -\frac{\pi(-1)^n}{n} + \frac{2}{n^3\pi}\left[(-1)^n - 1\right]$$

Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{\pi^2}{6} + 2\left[-\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \ldots\ldots\right]$$

$$-\pi\frac{(-1)^n}{n}+\frac{2}{\pi}(-2)\sin x+\frac{2}{3^3\pi}[-2]\sin 3x+\ldots\ldots$$

Q. 2. (a) Using Fourier Integral representation show that :

$$\int_0^\infty \frac{\cos x\alpha+\alpha\sin x\alpha}{1+\alpha^2}d\alpha=\begin{cases}0 & \text{if } x<0\\ \frac{\pi}{2} & \text{if } x=0\\ \pi e^{-x} & \text{if } x>0\end{cases}.$$

Ans.

$$f(x)=\begin{cases}x & , \quad 0<x<\pi/2\\ 0 & , \quad \pi/2<x<\pi\end{cases}$$

$$a_0=\frac{2}{\pi}\int_0^\pi f(x)\,dx=\frac{2}{\pi}\left[\int_0^{\pi/2}x\,dx+\int_{\pi/2}^{\pi}0\,dx\right]$$

$$=\frac{2}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi/2}\right]=\frac{1}{\pi}\left[\frac{\pi^2}{4}\right]=\frac{\pi}{4}$$

$$a_n=\frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$$

$$=\frac{2}{\pi}\left[\int_0^{\pi/2}x\cos nx\,dx+\int_{\pi/2}^{\pi}0\cos nx\,dx\right]$$

$$=\frac{2}{\pi}\left[x\frac{\sin nx}{n}+(1)\left(+\frac{\cos nx}{n^2}\right)\right]_0^{\pi/2}$$

$$=\frac{2}{\pi}\left[\frac{\pi}{2n}\sin\frac{n\pi}{2}+\frac{1}{n^2}\cos\frac{n\pi}{2}-\frac{1}{n^2}\right]$$

$$=\frac{2}{\pi}\left[\frac{\pi}{2}\sin\frac{n\pi}{2}+\frac{1}{n^2}\left[\cos\frac{n\pi}{2}-1\right]\right]$$

$$=\frac{1}{n}\sin\frac{2n\pi}{2}+\frac{2}{n^2\pi}\left[\cos\frac{n\pi}{2}-1\right]$$

$$b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx\,dx$$

$$=\frac{2}{\pi}\left[\int_0^{\pi/2}x\sin nx\,dx+\int_{\pi/2}^{\pi}0\,dx\right.$$

$$=\frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)+1\left(+\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2}$$

$$=\frac{2}{\pi}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2}+\frac{1}{n^2}\sin\frac{n\pi}{2}\right]=-\frac{1}{4}\cos\frac{n\pi}{2}+\frac{2}{n^2\pi}\sin\frac{n\pi}{2}$$

Fourier sine series,

$$=\sum_{n=1}^{\infty}b_n\sin nx$$

$$=-0+\frac{2}{\pi}(1)+\left(+\frac{1}{2}\right)+\frac{2}{2^2\pi}(0)-\frac{1}{3}(0)+\frac{2}{3^2\pi}(-1)$$

$$-\frac{1}{4}(1)+(0)-\frac{1}{5}(0)+\frac{2}{5^2\pi}(1)+\ldots\ldots$$

$$=\frac{2}{\pi}+\frac{1}{2}+\frac{2}{3^2\pi}(-1)-\frac{1}{4}+\frac{2}{5^2\pi}(1)+\ldots..$$

Cosine series is,

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$$

$$= \frac{\pi}{8} + \frac{1}{1} + \frac{2}{1^2\pi}(-1) + 0 + \frac{2}{2^2\pi}[-2] + \frac{1}{3}(1) + \frac{2}{3^2\pi}[-2] + \ldots$$

$$= \frac{\pi}{8} + 1 + \frac{2}{1^2\pi}(-1) + \frac{2}{2^2\pi}(-2) + \frac{1}{3} + \frac{2}{3^2\pi}(-2) + \ldots\ldots$$

Q. 2. (b) Find the inverse Fourier Transform of :

$$\frac{1}{s}e^{-as}.$$

Ans. By sine inversion formula,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(s)\sin sx dx$$

$$= \frac{2}{\pi}\int_0^\infty \frac{e^{-as}}{s}\sin sx\, ds \qquad \ldots(1)$$

$$\frac{df}{dx} = \frac{2}{\pi}\int_0^\infty e^{-as}\cos sx\, ds = \frac{2}{\pi}\frac{a}{x^2+a^2} \text{ by L.T.}$$

Integrating,

$$f(x) = \frac{2}{\pi}\int\frac{a dx}{x^2+a^2} = \frac{2}{\pi}\tan^{-1}\left(\frac{x}{a}\right) + c \qquad \ldots(2)$$

When $x = 0$, $f = 0$ by (1)

$\Rightarrow$ from (2) $c = 0$

$$f(x) = \frac{2}{\pi}\tan^{-1}\frac{x}{a}$$

Part-B

Q. 3. (a) Prove that :

(i) $\overline{\sin z} = \sin \bar{z}$

(ii) $\overline{\cos z} = \cos \bar{z}$

(iii) $\overline{\tan z} = \cos \bar{z}$

Ans. (i) $\overline{\sin z} = \sin \bar{z}$:

LHS : $\overline{\sin(x+iy)} = \overline{\sin x \cos iy + \cos x \sin iy}$

$$= \overline{\sin x \cosh y + i \cos x \sinh y}$$

$$= \sin x \cosh y - i \cos x \sinh y$$

RHS $\sin \bar{z} = \sin(x - iy)$

$$= \sin x \cos iy - \cos x \sin iy$$

$$= \sin x \cosh y - \cos x i \sinh y$$

LHS= RHS.

(ii) $\overline{\cos z} = \cos \bar{z}$:

LHS : $\overline{\cos(x+iy)} = \overline{\cos x \cos iy - \sin x \sin iy}$

$$= \overline{\cos x \cosh y - i \sin x \sinh y}$$

$$= \cos x \cosh y + i \sin x \sinh y$$

R.H.S.

$$\cos \bar{z} = \cos(x - iy)$$

$$= \cos x \cos iy + \sin x \sin iy$$

$$= \cos x \cosh y + \sin x \, i \sinh y$$

LHS = RHS.

(iii) $\overline{\tan z} = \cos \bar{z}$:

LHS $\overline{\tan(x+iy)}$

$$\frac{\tan x + \tan iy}{1 - \tan x \sin iy} = \frac{\tan + \frac{\sin iy}{\cos iy}}{1 - \tan x \frac{\sin iy}{\cos iy}}$$

$$= \frac{\frac{\sin x}{\cos x} + \frac{\sin iy}{\cos iy}}{1 - \frac{\sin x}{\cos x}\frac{\sin iy}{\cos iy}} = \frac{\sin \cos iy + \sin iy \cos x}{\cos x \cos iy - \sin ix \sin iy}$$

$$= \frac{\sin x \cosh y + i \sinh y \cos x}{\cos x \cosh y - i \sin x \sinh y}$$

$$= \frac{\sin x \cosh y + i \sinh y \cos x}{\cos x \cosh y - i \sin x \sinh y} \times \frac{\cos x \cosh y + i \sin ix \sinh y}{\cos x \cosh y + i \sin ix \sinh y}$$

$$= \frac{(\sin x \cos x \cosh^2 y - \sin x \cos x \sin^2 y) + i(\sinh y \cos^2 x \cosh y + \sin^2 x \cosh y \sinh y)}{(\cos x \cosh y)^2 + (\sin x \sinh y)^2}$$

$$= \frac{\sin x \cos x(\cosh^2 y - \sinh^2 y) + i \sinh y \cosh y(\cos^2 x - \sin^2 x)}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^3 y}$$

$$= \frac{\sin x \cos x(1) + i \sinh y \cosh y\left(1 - 2\sin^2 x\right)}{\cos^2 x\left(1 + \sin^2 y\right) + \sin^2 x \sinh^2 hy}$$

Q. 3. (b) Show that the function :

$$f(z) = \begin{cases} \dfrac{\left(x^3 - y^3\right) + i\left(x^3 + y^3\right)}{x^2 + y^2} & , \quad z \neq 0 \\ 0 & , \quad z = 0 \end{cases}$$

satisfies C-R equations at the origin but does not have a derivative at origin.

Ans. Here,

$$f(z) = \frac{\left(x^3 - y^3\right) + i\left(x^3 + y^3\right)}{x^2 + y^2}, \; z \neq 0$$

Let $f(z) = u + iv$ then

$$u = \frac{x^3 - y^3}{x^2 + y^2}, \quad v = \frac{x^3 + y^3}{x^2 + y^2}$$

Since $z \neq 0 \Rightarrow x \neq 0, y \neq 0$.

$\therefore$ u & v are rational function of x and y with non-zero denominators. Thus, u, v and hence f(z) are continuous functions when $z \neq 0$. To test them for continuity at $z = 0$, an changing u, v to polar co-ordinates by putting $x = r\cos\theta$, $y = r\sin\theta$, we get

$$u = r\left(\cos^3\theta - \sin^3\theta\right)$$

and $$v = r\left(\cos^3\theta + \sin^3\theta\right)$$

When $z \to 0$, $r \to 0$

$\therefore \qquad \lim_{z\to 0} = u \lim_{z\to 0} r\left(\cos^3\theta - \sin^3\theta\right) = 0$

By $\lim_{z\to 0} v = 0 \;\therefore\; \lim_{z\to 0} f(z) = 0 = f(0)$

$\Rightarrow$ f(z) is continuous at z = 0.

Hence, f(z) is continuous for all values of z. At the origin (0,), we have

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0} \frac{x-0}{x} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y\to 0} \frac{-y-0}{x} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x\to 0} \frac{-y-0}{y} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y\to 0} \frac{y-0}{y} = 1$$

$\therefore \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

and $\qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Hence, CR equations are satisfied at origin.

Now,

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z} \lim_{z\to 0} \frac{\left(x^3 - x^3\right) + i\left(x^3 + y^3\right)}{\left(x^2 + y^2\right)(x+y)}$$

Let $z \to 0$ along the line y = x then

$$f'(0) = \lim_{x \to 0} \frac{0 + 2ix^3}{2x^3(1+i)} = \frac{i}{1+i} = \frac{i(1-i)}{2} = \frac{1+i}{2} \quad ...(2)$$

Also, let $z \to 0$ along the x axis $(y = 0)$. Then

$$f'(0) = \lim_{x \to 0} \frac{x^3 + ix^3}{x^3} = 1 + i \quad ...(2)$$

Since the limits (1) and (2) are different

$\Rightarrow f'(0)$ does not exist.

Q. 4. (a) Evaluate $\int_C (z - z^2)\,dz$ where C is the upper half of the circle $|z - 2| = 3$. What is the value of the integral if C is the lower half of the above given circle?

Ans. $\int (z - z^2)\,dz$

As $|z - 2| = 3$

$\Rightarrow$ $z - 2 = 3e^{i\theta}$

$z = 2 + 3e^{i\theta}$

$dz = 3ie^{i\theta}d\theta$

Upper half circle :

$$\int (z - z^2)\,dz = \int_0^{\pi} \left[\left(2 + 3e^{i\theta}\right) - \left(2 + 3e^{i\theta}\right)^2 \right] 3ie^{i\theta} d\theta$$

$$= \int_0^{\pi} \left[\left(2 + 3e^{i\theta}\right) - \left(4 + 9e^{2i\theta} + 12e^{i\theta}\right) \right] 3ie^{i\theta}\, d\theta$$

$$= \int_0^{\pi} \left(-2 - 9e^{i\theta} - 9e^{2i\theta}\right) 3ie^{i\theta}\, d\theta$$

$$= -i3\int_0^{\pi} 2e^{i\theta} + 9e^{2i\theta} + 9e^{3i\theta}\, d\theta$$

$$= -3i\left[\frac{2e^{i\theta}}{i} + \frac{9e^{2i\theta}}{2i} + \frac{9e^{3i\theta}}{3i}\right]_0^{\pi} \quad \text{..(A)}$$

$$= -3i\left[\left(2e^{i\pi} + \frac{9}{2}e^{2i\pi} + \frac{9}{3}e^{3i\pi}\right) - \left(2 + \frac{9}{2} + \frac{9}{3}\right)\right]$$

$$= -3\left[2(-1) + \frac{9}{2}(1) + \frac{9}{3}(-1) - 2 - \frac{9}{2} - \frac{9}{3}\right]$$

$$= -3[-4-6] = 30$$

In lower half circle (limit beam π to 2π) by using (A)

$$-3\left[2e^{i\theta} + \frac{9}{2}e^{2i\theta} + \frac{9}{3}e^{3i\theta}\right]_{\pi}^{2\pi}$$

$$-3\left[\left(2(1) + \frac{9}{2}(1) + \frac{9}{3}(1)\right) - \left(2(-1) + \frac{9}{2}(1) + \frac{9}{3}(-1)\right)\right]$$

$$-3(4+6) = -30$$

It because -30 in lower half circle.

Q. 4. (b) Expand $\frac{1}{(z+1)(z+3)}$ in Laurent series valid for :

(i) $1 < |z| < 3$

(ii) $0 < |z+1| < 2$

(iii) $|z| > 2$.

Ans. $\dfrac{1}{(z+1)(z+3)}$

$$= \frac{A}{z+1} + \frac{B}{z+3}$$

$$1 = A(z+3) + B(z+1)$$

Put $z = -1$

$$1 = A(2) \qquad \boxed{A = \frac{1}{2}}$$

Put $z = -3$

$$1 = -2B \Rightarrow \boxed{B = -\frac{1}{2}}$$

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$$

(i) $1 < |z| < 3$ **;**

$$\frac{1}{2}\left[\frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{1}{3\left(1+\frac{2}{3}\right)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{3}\left(1+\frac{z}{3}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \frac{1}{z^4} + \ldots\ldots\right) + \frac{1}{6}\left(1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \ldots\right)\right]$$

$$f(z) = + = \frac{1}{2z^4} + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \dots\dots$$

(ii) $0 < |z+1| < 2$:

$$= \frac{1}{2(z+1)} - \frac{1}{(z+1+2)}$$

$$= \frac{1}{2}(z+1)^{-1} - \left[\frac{1}{2\left(1+\frac{z+1}{2}\right)}\right]$$

$$= \frac{1}{2}(z+1)^{-1} - \frac{1}{2}\left(1+\frac{z+1}{z}\right)^{-1}$$

$$= \frac{1}{2}(z+1)^{4} - \frac{1}{2}\left[1-\left(\frac{z+1}{2}\right)+\left(\frac{z+1}{2}\right)^2-\left(\frac{z+1}{2}\right)^3+\left(\frac{z+1}{2}\right)^4+\dots\dots\right].$$

(iii) $|z| > 2$:

$$= \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{1}{z\left(1+\frac{3}{z}\right)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{z}\left(1+\frac{3}{z}\right)^{-1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\frac{1}{z^4}+\ldots\right)-\frac{1}{z}\left[1-\frac{3}{2}+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^3+\ldots\right]\right]$$

$$= \frac{1}{2z}\left[\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\frac{1}{z^4}+\ldots\right)-\left(1-\frac{3}{z}+\left(\frac{3}{z}\right)^2-\left(\frac{3}{2}\right)^3\right)+\ldots\right]$$

$$= \frac{1}{z^2}-\frac{4}{z^3}+\frac{13}{z^4}-\ldots\ldots$$

Q. 5. (a) Evaluate $\int_C \frac{e^{3z}}{(z-\ln 2)^4}$ where C is the square with vertices at $\pm 1, \pm i$.

Ans.

$$\oint_C \frac{e^{3z}}{(z-\log 2)^4}dz$$

The integrand has a singularity at $z=\log 2$ which lies within the square.

Now,

$$f^n(a) = \frac{\lfloor n}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz$$

Here,

$a=\log 2$, $n+1=4$ is $n=3$, $f(z)=e^{3z}$

$$f^n(z)=3^n e^{3z},$$

$$f'''(z)=3^3 e^{3z}$$

$$f'''(\log 2)=3^3 e^{3\log 2}=3^3 e^{\log 2^3}=3^3 2^3$$

$$f'''(\log 2) = \frac{\lfloor 3}{2\pi i}\int_C \frac{e^{3z}}{(z-\log 2)^4}\,dz$$

$$3^2 2^3 = \frac{\lfloor 3}{2\pi i}\int_C \frac{e^{3z}}{(z-\log 2)^4}\,dz$$

$$\frac{(27z^3)2\pi i}{6} = \oint \frac{e^{3z}}{(z-\log 2)^4}\,dz$$

$$(9z^3)\pi i = \oint \frac{e^{3z}}{(z-\log 2)^4}\,dz$$

$$\Rightarrow \quad \boxed{\oint \frac{e^{3z}}{(z-\log 2)^4} = 8\pi i z^3 = 72\pi i} \qquad \text{Ans.}$$

Q. 5. (b) Evaluate :

$$\int_0^\infty \frac{dx}{(a^2+x^2)^2}.$$

Ans. Let $\displaystyle\int_0^\infty \frac{dx}{(a^2+x^2)^2}$

Poles of $\phi(z) = \dfrac{1}{(a^2+z^2)^2}$ are obtained by solving by solving $a^2+z^2=0$

$$z = \pm ia,\ \pm ia$$

Residue of f(z) at $z = ia$ is

$$\frac{1}{\lfloor 1}\lim_{z\to ia}\left[\frac{d}{dz}(z-ia)^2\frac{1}{(z+ia)^2(z-ia)^2}\right]$$

$$=\lim_{z\to ia}\left[\frac{d}{dz}\frac{1}{(z+ia)^2}\right]$$

$$=\lim_{z\to ia}\left[-2(z+ia)^{-3}\right]=-2(2ia)^{-3}$$

$$=\frac{-2}{8i^3a^3}$$

$$=\frac{1}{4a^3 i}\frac{i}{i}$$

$$=\frac{i}{4a^3}\frac{1}{i^2}$$

$$=-\frac{i}{49^3}$$

$$=-\frac{i}{49a^3}$$

By residue of f(z) at $z = -ia$ is

$$\frac{1}{\lfloor 1}\lim_{z\to ia}\left[\frac{d}{dz}(z+ia)^2\frac{1}{(z+ia)^2(z-ia)^2}\right]$$

$$\Rightarrow \lim_{z\to ia}\left[\frac{d}{dz}(z+ia)^2\right]$$

$$\lim_{z \to ia} -2(z-ia)^{-3} = -2(-2ia)^{-3}$$

$$= \frac{-2}{-8i^3a^3}$$

$$= -\frac{1}{4ia^3}$$

$$= +\frac{i}{4a^3}$$

By Residue theorem,

$$\int_0^{\infty} \frac{dx}{\left(a^2+x^2\right)^2} = 2\pi i\left[-\frac{i}{4a^2}+\frac{Ii}{4a^3}\right]$$

Part-C

Q. 6. (a) A box contains 9 tickets numbered 1 to inclusive. If 3 ticket are drawn from the box one at a time, find the probability they are alternatively either odd, even, odd or even, odd, even.

Ans. Because box contain 9 tickets number 1 to 9

Even numbers are 2, 4, 6, 8

Odd numbers are 1, 3, 5, 7, 9

If they are alternating in the order (A)

Odd even odd

Then the probability is

$$\frac{{}^5C_1}{{}^9C_1} \times \frac{{}^4C_1}{{}^8C_1} \times \frac{{}^4C_1}{{}^7C_1}$$

$$= \frac{5}{9} \times \frac{4}{8} \times \frac{4}{7} = \frac{10}{63}$$

If they are alternatively in order B (even odd even)

$$\frac{{}^4C_1}{{}^9C_1} \times \frac{{}^5C_1}{{}^8C_1} \times \frac{{}^3C_1}{{}^7C_1}$$

$$= \frac{4}{9} \times \frac{5}{8} \times \frac{3}{7} = \frac{5}{42}$$

$\Rightarrow$ Probability they are alternatively either odd, even odd or even, odd, even is,

$$= \frac{10}{63} + \frac{5}{42}$$

$$= 0.1587 + 0.119 = 0.27774.$$

Q. 6. (b) Fit a binomial distribution to the following data :

x :	0	1	2	3	4
f :	30	62	46	10	2

Ans.

x	f	f(x)
0	30	0
1	62	62
2	46	92
3	10	30
4	02	08
$N = \Sigma f = 150$		

$$\Sigma fx = 192$$

$$\text{Mean} = \frac{\Sigma f(x)}{\Sigma f} = \frac{192}{150}$$

$$np = 1.28$$

$$p = \frac{1.28}{4} = .32$$

$$q = .68$$

Hence, the binomial distribution to be fitted to data is,

$$150(.32+.68)^4$$

Theoretical frequencies

X	N	${}^nC_r\, p^r q^{n-r}$
0	150	${}^4C_0 (.32)^0 (.68)^4$
1		$150\,{}^4C_1 (.32)^1 (.68)^3 1$
2	$150\,{}^4C_1 (.32)^1 (.68)^3$	
3	$150\,{}^4C_3 (.32)^3 (.68)^1$	
4	$150\,{}^4C_4 (.32)^4 (.68)^0$	

Q. 7. (a) Mice with an average life span of 32 months will live upto 40 months when fed by a certain nitrous food. If 64 mice fed on this diet have an average life span of 38 months and standard deviation of 5.8 months, is there any reason to believe that average life span is less than 40 months.

(b) Test for goodness of fit of a Poisson distribution at 0.05 level of significance to the following frequency distribution :

No. of patients arriving/hour (x) :	0	1	2	3	4	5	6	7	8
Frequency :	52	151	130	102	45	12	5	1	2

$$\left(\chi^2_{0.05} = 14.067 \text{ with } v = 7\right)$$

Ans. Mean of the given distribution is,

$$\bar{x} = \frac{\Sigma f_x}{\Sigma f} = \frac{0\times52+1\times151+2\times130+3\times102+4\times45+5\times129+6\times5+7\times1+8\times2}{52+151+130+102+45+12+5+1+2}$$

$$= \frac{0+151+260+306+180+60+30+7+16}{500}$$

$$= \frac{1010}{500} = 2.02$$

In order to fit a poison distribution to the given data, we take mean no. of poisson distribution equal to the mean of the given distribution i.e., $m = \bar{x} = 2.02$.

The theoretical frequency are given by

$$f(r) = \frac{N \times e^{-2.02}(2.02)^r}{|r}$$

$$f(0) = \frac{500 \times 0.135}{\lfloor 0} = 67.5 \quad r = 0, 1, 2, 3, \ldots\ldots\ldots r$$

$$f(1) = \frac{500 \times 2.02 \times 0.135}{\lfloor 1} = 136.67$$

$$f(2) = \frac{500 \times 0.135 \times (2.02)}{\lfloor 2} = 137.71$$

$$f(3) = \frac{500 \times 0.135 \times (2.02)^3}{\lfloor 3 = 6} = 92.71$$

$$f(4) = \frac{500 \times 0.135 \times (2.02)^4}{\lfloor 4 = 24} = 46.82$$

$$f(5) = \frac{500 \times 0.135 \times (2.02)^5}{120} = 18.91$$

$$f(6) = \frac{500 \times 0.135 \times (2.02)^6}{\lfloor 6} = 6.36$$

$$f(7) = \frac{500 \times 0.135 \times (2.02)^7}{\lfloor 7} = 1.83$$

$$f(8) = \frac{500 \times 0.135 \times (2.02)^8}{\lfloor 8} = 0.46$$

Theoretical Poisson frequency correct to one decimal place are.

X	1	2	3	4	5	6	7	8
Expected	67.5	136.6	137.7	92.7	46.8	18.9	6.3	1.8 0.46

Calculation of chi square

Observed Frequency (0)	Expected Frequency E	$(O-E)^2$	$(O-E)^2/K$
52	67.5	225	3.35
151	136.6	225	1.65
130	137.7	49	0.35
102	92.7	100	1.08
45	46.8	1	0.02
12	18.9	36	2
5	6.3	1	0.166
1	1.8	0	0
2	0.4	2.56	6.4

$$\chi^2 = \sum \frac{(O-E)^2}{E} = 15.016$$

Given $\chi^2_{0.05} = 14.067$ with $v = 7$.

Conclusion : Since the calculated value is close to 15.016. Hence we c[illegible] that Poisson distribution is fit to the given data. (It is highly significant).

Q. 8. (a) Solve the following L.P.P. graphically :

Maximize $Z = 3x_1 + 4x_2$

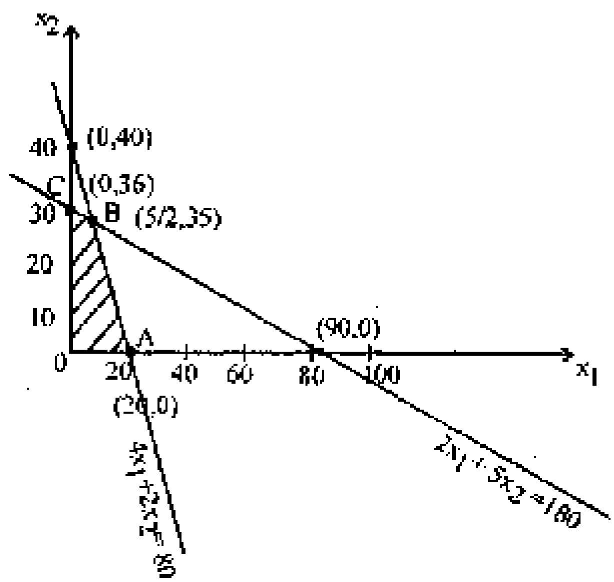
Subject to :

$4x_1 + 2x_2 \le 80,$

$2x_1 + 5x_2 \le 180$

$x_1, x_2 \ge 0.$

Ans.



Max $\quad Z = 3x_1 + 4x_2$

Sub. to $\quad 4x_1 + 2x_2 \leq$ [illegible]

$2x_1 + 5x_2 \leq 180$

$x_1, x_2 \geq 0$

Corresponding equality is

$4x_1 + 2x_2 = 80 \rightarrow$

x_1	0	20
x_2	40	0

$2x_1 + 5x_2 = 180 \rightarrow$

Optimal [illegible]

$O(0,0); A(20,0), B\left(\frac{5}{2}, 35\right), C(0, 36)$

$Z_O = 0$

$Z_A = 3(20) + 4(0) = 60$

$$Z_B = 3\left(\frac{5}{2}\right) + 4(35) = \frac{15}{2} + 140 = 147.5$$

$$Z_C = 3(0) + 4(36) = 144$$

Max. value at B (5/2, 35)

$$\boxed{Z_B = 147.5}.$$

Q. 8. (b) Using Dual Simplex Method :

Maximize $\quad Z = -3x_1 - x_2$

Subject to,

$$x_1 + x_2 \geq 1,$$

$$2x_1 + 3x_2 \geq 2$$

$$x_1, x_2 \geq 0.$$

Ans. Maximize $\quad Z = -3x_1 - x_2$

Subject to,

$$x_1 + x_2 \geq 1,$$

$2x_1 + 3x_2 \geq 2$

$x_1 , x_2 \geq 0$

Convert the "$\geq$" type constant to "$\leq$" type above L.P.P. takes the form.,

Max., $Z = -3x_1 - x_2 + 0s_1 + 0s_2$

Sub to : $-x_1 - x_2 \leq -1$

$-2x_1 - 3x_2 \leq -2$

Convert in equations into equation by adding slack variable.

$-x_1 - x_2 + s_1 = -1$

$-2x_1 - 3x_2 + s_2 = -2$

		C_j	-3	-1	0	0
C_B	Basic variable	b	x_1	x_2	s_1	s_2
0	s_1	-1	-1	-1	1	0

←	0	s_2	-2	-2	$\boxed{-3}$	0	1
				↑			

	$\Delta_j =$	$C_j - Z_5 =$	-3	-1	0	0

	C_j	-3	-1	0	0	
C_B	Basic	b	x_1	x_2	s_1	s_2
← 0	s_1	-1/3	$\boxed{1/3}$	0	1	-1/3
-1	x_2	2/3	2/3	1	0	-1/3
			↑			

$\Delta_j = C_j - Z_j$ -7/3 0 0 1/3

$$\text{Mini.}\left(\frac{-7/3}{1/3}, \frac{1/3}{-1/3}\right) = -7$$

		C_j	-3	-1	0	0
C_B	Basic	b	x_1	x_2	s_1	s_2

–3	x_1	–1	1	0	3	–1
–1	x_2	4/3	0	1	–2	1/3
	$\Delta j = C_j - Z_5$	0	0	–7	–8/3	

$\because$ All $\Delta_j \le 0$ but b_j is –ve, so this problem has unbounded solution.