B.E.
Third Semester Examination, Dec-2008
MATHEMATICS-III

Note : Attempt any five questions, selecting at least one question from each part.

Part-A

Q. 1. (a) Expand $f(x) = x\sin x$, $0 < x < 2\pi$ as a Fourier series.

Ans. $$f(x) = x\sin x \qquad 0 < x < 2\pi$$

Now, $$f(x) = x\sin x$$

$$f(-x) = -x\sin(-x)$$

$$= x\sin x$$

Means its even function,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \frac{\cos n\pi x}{C}$$

$$a_0 = \frac{2}{2\pi}\int_0^{2\pi} x\sin x\,dx$$

$$= \frac{1}{\pi}\left[x(-\cos x) - \int(-\cos x)dx\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x\cos x + \sin x\right]_0^{2\pi} = -\frac{1}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos\frac{nx}{2}dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin\left(\frac{nx}{2}+1\right) - \sin\left(\frac{nx}{2}-x\right)\right]dx$$

$$= \frac{1}{2\pi}\left[x\left\{\frac{1}{n/2+1}\left[-\cos\left(\frac{n}{2}+1\right)\right]\right] + \frac{1}{n/2-1}\left(\frac{n}{2}-1\right)x\right\}$$

$$-\int\left\{\frac{1}{n/2+1}\left[-\cos\left(\frac{n}{2}+1\right)\right]+\frac{1}{n/2-1}\cos\left(\frac{n}{2}-1\right)x\right\}dx\Bigg]_0^{2\pi}$$

$$=\frac{1}{2\pi}\left[x\left\{-\frac{\cos(n/2+1)x}{n/2+1}+\frac{\cos(n/2-1)}{n/2-1}x\right\}+\frac{\sin(n/2+1)x}{(n/2+1)^2}-\frac{\sin(n/2-1)x}{(n/2-1)^2}\right]_0^{2\pi}$$

$$=\frac{1}{2\pi}\left[2\pi\left\{\frac{\cos(n/2-1)2\pi}{(n/2-1)}-\frac{\cos(n/2+1)2\pi}{(n/2+1)}\right\}+\frac{\sin(n/2+1)2\pi}{(n/2+1)^2}-\frac{\sin(n/2-1)2\pi}{(n/2-1)^2}\right]$$

$$f(x)=-\frac{1}{\pi}+\frac{1}{2\pi}\left[(-2\pi+2)+\left(\frac{16\pi}{3}\right)+\ldots\ldots\right].$$

Q. 1. (b) Obtain a half range series for

$$f(x)=kx \text{ for } 0\le x\le\frac{\ell}{2}$$

$$=k(\ell-x) \text{ for } \frac{\ell}{2}\le x\le\ell$$

Deduce the sum of the series

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\ldots.$$

Ans. $f(x)=kx$ for $0\le x\le 1/2$

$$a_1=\frac{4}{1}\int_0^{1/2}f(x)\,dx=\frac{4}{1}\int_0^{1/2}kx\,dx$$

$$=\frac{4}{1}\left[\frac{kx^2}{2}\right]_0^{1/2}$$

$$=\frac{2}{1}\left[\frac{kx^2}{2}\right]=\frac{kl}{2}$$

$$a_2=\frac{2}{1}\int_{1/2}^{1}k(1-x)\,dx$$

$$= \frac{2k}{l}\left[lx - \frac{x^2}{2}\right]_{l/2}^{l}$$

$$= \frac{2k}{l}\left[l.l - \frac{l^2}{2} - \frac{l.l}{2} + \frac{l^2}{8}\right]$$

$$= \frac{2k}{l}\left[\frac{l^2}{8}\right] = \frac{kl}{4}$$

$$f(x) = \frac{kl}{2} + \frac{kl}{4}$$

$$= \frac{k}{2}[1+2+\ldots\ldots] + \frac{k}{4}[1+2+\ldots\ldots]$$

$$f(x) = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

$$= \frac{1}{(2n+1)^2}$$

Q. 2. (a) State and prove convolution theorem for Fourier transforms.

Ans. The convolution of two functions $f(x)$ *and* $g(x)$ is defined as,

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(u) g(x-u)\, dx$$

Convolution theorem on Fourier transform. The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transform. i.e.,

$$F[f(x) * g(x)] = F[f(x)].F[g(x)]$$

Proof : We know that

$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u).g(x-u)\, du$$

Taking Fourier transform of both sides of (1). We have

$$F[f(x) * g(x)] = F\left[\int_{-\infty}^{\infty} f(u).g(x-u).du\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u).g(x-u)du \right] e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)du \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{\infty} g(x-u)e^{isx} dx \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \{ f(u).du.Fg(x-u) \}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)du.e^{ius} G(s) \quad \text{(using shifting property)}$$

$$= G(s). \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)e^{ius} du$$

$$= G(s).F(s)$$

By inversion

$$F^{-1}\{F(s).G(s)\} = f * g = F^{-1}\{F(s) * F^{-1}\{G(s)\}\}.$$

Q. 2. (b) Find the Fourier sine transform of

$$\frac{1}{x(x^2+a^2)}.$$

Ans. Fourier sine transform of

$$F(x) = \frac{1}{x(x^2+a^2)}$$

$$F_s\{f(x)\} = \int_0^{\infty} f(x) \sin sx \, dx$$

$$= \int_0^{\infty} \frac{1}{x(x^2+a^2)} \sin sx \, dx = F(s)$$

Differentiating both sides w.r.t. S, we get

$$\frac{d}{ds}\{F(s)\} = \int_0^\infty \frac{x \cos sx}{x(x^2+a^2)}$$

$$= \int_0^\infty \frac{\cos sx}{(x^2+a^2)} = \frac{a}{s^2+a^2}$$

Integration w.r.t. S, we obtain,

$$F(s) = \int \frac{a}{s^2+a^2} ds$$

$$= \tan^{-1}\frac{s}{a} + c$$

But $F(s) = 0$, when $s = 0$

$\therefore$ $c = 0$

Hence, $F(s) = \tan^{-1}\left(\frac{s}{a}\right)$.

Part-B

Q. 3. (a) If $\cosh x = \sec\theta$, prove that

$$\tanh^2\frac{x}{2} = \tan^2\frac{\theta}{2}.$$

Ans. If $\cosh x = \sec\theta$

Prove that $\tan^2\frac{x}{2} = \tan^2\frac{\theta}{2}$

We have $e^u = \tan\left(\frac{\pi}{4}+\frac{P}{2}\right)$

Or $$\frac{e^{u/2}}{e^{-u/2}} = \frac{1+\tan\theta/2}{1-\tan\theta/2}$$

By components & dividendo, we get

$$\frac{e^{u/2}-e^{-u/2}}{e^{u/2}+e^{-u/2}} = \tan\frac{\theta}{2}$$

i.e., $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$

$$\frac{1}{i}\tan\frac{iu}{2} = \frac{1}{i}\tanh\frac{i\theta}{2}$$

Or
$$\frac{i\theta}{2} = \tanh^{-1}\left(\tan\frac{iu}{2}\right)$$

$$= \frac{1}{2}\log\frac{1+\tan iu/2}{1-\tan iu/2}$$

By taking Antilog both sides

$$\tan^2\frac{x}{2} = \tan^2\frac{\theta}{2}.$$

Q. 3. (b) Reduce $\tan^{-1}(\cos\theta + i\sin\theta)$ to the form a+ib. Hence show that

$$\tan^{-1}\left(e^{i\theta}\right) = \frac{n\pi}{2} + \frac{\pi}{4} - \frac{i}{2}\log\tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right).$$

Ans. $\tan^{-1}(\cos\theta + i\sin\theta)$

Let $\tan^{-1}(\cos\theta + i\sin\theta) = x + iy$

Then $\cos\theta + i\sin\theta = \tan(x+iy)$

$$= \tan x\,\mathrm{sech}\,y + i\sec x\tanh y$$

$\therefore$ $\cos\theta = \tan x\,\mathrm{sech}\,y$ ...(1)

& $\sin\theta = \sec x\tanh y$ ...(2)

Squaring & adding equation (1) & (2)

$$1 = \tan^2 x(\mathrm{sech}\,y)^2 + \sec^2 x(\tanh y)^2$$

$$= \sin^2 x + \sinh^2 y\left(\sin^2 x + \cos^2 x\right)$$

$$1 - \sin^2 x = \sinh^2 y$$

i.e., $\cos^2 x = \sinh^2 y$

Now, $\tan^{-1}\left(e^{i\theta}\right) = \tan\mathrm{sech}\,y + \sec x\tanh y$

$$\tan^{-1}\left(e^{-i\theta}\right) = \frac{n\pi}{2} + \frac{\pi}{4} - \frac{i}{2}\log\left(\frac{\pi}{2} + \frac{\theta}{2}\right).$$

Q. 4. (a) State and prove C-R equations and show that these are necessary for a function to be analytic in a region.

If real part of an analytic function is

$$x^3 - 3xy^2 + 3x^2 - 3y^2 + 1,$$

find the function f(z).

Ans. Theorem : The necessary and sufficient conditions for the derivative of the $w = u(x,y) + iv(x,y) = f(z)$ to exist for all values of z in a region R, are

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y in R;

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

The relations (ii) are known as Cauchy-Riemann equations or briefly C-R equation

(a) Condition is necessary :

Let δu and δv be the increments of u and v respectively corresponding to the increments δx and δy of x and y, so that $\delta z = \delta x + i\delta y$.

If f(z) possesses a unique derivative at P(z), then

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z \to 0} \frac{(u+\delta u) + i(v+\delta v) - (u+iv)}{\delta z} = \lim_{\delta z \to 0}\left(\frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}\right)$$

Since δz can approach zero in any manner, we can first assume δz to be wholly real wholly imaginary. When δz is wholly real, then $\delta y = 0$ and $\delta z = \delta x$.

$$\therefore \quad f'(z) = \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

When δz is wholly imaginary, then $\delta x = 0$ and $\delta z = i\delta y$.

$$\therefore \quad f'(z) = \lim_{\delta y \to 0}\left(\frac{\delta u}{i\delta y} + i\frac{\delta v}{i\delta y}\right) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} - i\frac{\partial u}{\partial y}$$

Now the existence of f'(z) requires the equality of (1) and (2).

$$\therefore \quad \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \qquad ...(2)$$

On equating the real and imaginary parts from both sides, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad ...(3)$$

Thus, the necessary conditions for the existence of the derivative of f(z) is that the C-R equations should be satisfied.

(b) Condition is sufficient :

Suppose f(z) is a single-valued function possessing partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at each point of the region and the C-R equations (3) are satisfied.

Then by Taylor's theorem for functions of two variables (p. 176).

$$f(z+\delta z) = u(x+\delta x, y) + iv(x+\delta x, y+\delta y)$$

$$= u(x,y) + \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y\right) + ...+ i\left(v(x,y) + \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right) + ...\right)$$

$$= f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial u}{\partial y}\right)\delta y$$

[Omitting terms beyond the first powers of δx and δy]

$$f(z+\delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

Now using the C-R equations (3), replace $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ by $-\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial x}$ respectively.

Then

$$f(z+\delta x) = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta x + \left[-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right]\delta y = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta x + \left[i\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right]i\delta y$$

$$= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right](\delta x + i\delta y) = \left[\frac{\partial u}{\partial x} + i\frac{\partial y}{\partial x}\right]\delta z$$

$$f'(z) = \lim_{\delta z \to 0}\frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \text{ or } \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

Real part of analytic function,

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

Let $$f(z) = u + iv$$

Where, $$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial v}{\partial y}$$

$$= 3x^2 - 3y^2 + 6x - i(-6y - 6xy)$$

$$= 3x^2 - 3y^2 + 6x + i(6y + 6xy)$$

By Milne-Thomson's method, we express $f'(z)$ in terms of z by putting $x = z$ & $y = 0$

$$\therefore \quad f'(z) = 3z^2 + 6z + i(0)$$

$$= 3z^2 + 6z$$

Integrating w.r.t z, we get

$$f(z) = \frac{3z^3}{3} + \frac{6z^2}{2}$$

$$= z^3 + 3z^2 + ic$$

$$f(z) = (z+3)z^2 + ic.$$

Q. 4. (b) Evaluate $\int_0^{2\pi} \frac{d\theta}{1 - 2a\sin\theta + a^2}$, $0 < a < 1$.

Ans. $$I = \int_0^{2\pi} \frac{d\theta}{1 - 2a\sin\theta + a^2}$$

Putting $z = e^{i\theta}$, $d\theta = dz/iz$,

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

And $$\sin\theta = \frac{1}{2}\left(z - \frac{1}{2}\right)$$

$$I = \int_C \frac{1}{1-2a\frac{1}{2}\left(z-\frac{1}{z}\right)+a^2}\frac{dz}{iz} = \frac{1}{i}\int_C \frac{z}{z-az^2+a+a^2iz^2}dz$$

$$= \frac{1}{i}\int_C f(z)dz \text{ where C is the unit circle}$$

Now f(z) has simple poles at $z = a, \frac{1}{a}$ and the second order pole at $z = 0$ of which the poles at $z = 0$ and $z = a$ lie within the unit circle,

$$\therefore \quad R_{es}f(a) = \lim_{z\to 0}\left[(z-a)f(z)\right]$$

$$= \frac{1}{i}\left[\frac{z}{z-az^2+a+a^2-iz^2}\right]$$

$$= \frac{1}{i}\frac{a}{\left(a-a^3+a+a^4i\right)}$$

$$\& \quad R_{es}\ f(0) = \lim_{z\to 0}\frac{d}{dz}\left[z^2 f(z)\right]$$

$$= \frac{1}{i}\lim_{z\to 0}\frac{d}{dz}\frac{z}{\left(z-az^2+a+a^2iz^2\right)}$$

$$= -\frac{1+a^2}{2ia^2}$$

Hence.

$$I = 2\pi i\left[\operatorname{Res} f(a) + \operatorname{Res} f(0)\right]$$

$$= 2\pi i\left[\frac{a}{i\left(a-a^3+a+a^4i\right)} - \frac{1+a^2}{2ia^2}\right].$$

Q. 5. (a) State and prove Residue theorem and use it to evaluate $\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)}dz$

Where C is the circle $|z| = 3$

Ans. Singular points of an analytic function :

We have already defined a singular point of a function as the point at which the function ceases to be analytic. If z = a is such a singular point of the function f(z) then there exists a circle with centre a which has no other singular point f(z), then z = a is called as isolated singular point. In such a case, f(z) can be expanded in a Laurent's series around z = a, giving

$$f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + \ldots\ldots + c_{-1}(z-a)^{-1} + c_{-2}(z-a)^{-2} + \ldots\ldots \qquad \ldots(1)$$

If all the negative powers of $(z-a)$ in (1) after the nth are missing, then the singular point z = a is called a pole of order n. A pole of first order is called a simple pole.

If the number of negative powers of (z-a) in (1) is infinite; then z = a is called as essential singularity.

2. Residues : The co-efficient of $(z-a)^{-1}$ in the expansion of f(z) around an isolated singularity is called the residue of f(z) at that point. Thus from (1), the residue f(z) at z = a is c_{-1}.

$\therefore$ $$\operatorname{Res} f(a) = \frac{1}{2\pi i}\int_C f(z)\,dz$$

i.e., $$\int_C f(z)\,dz = 2\pi i \operatorname{Res} f(a)$$

Residue theorem : If f(z) is analytic in a closed curve C except at a finite number of singular points within C, then

$$\int_C f(z)\,dz = 2\pi i \times \text{(sum of the residues at the singular points within C)}$$

Let us surround each of the singular points $a_1, a_2, \ldots\ldots, a_n$ by a small circle such that it encloses no other singular point. Then these circles $C_1, C_2, \ldots\ldots, C_n$ together with C, form a multiply connected region in which f(z) is analytic.

$\therefore$ Applying Cauchy's theorem, we have

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \ldots\ldots + \int_{C_n} f(z)\,dz$$

$$= 2\pi i\left[\operatorname{Res} f(a_1) + \operatorname{Res} f(a_2) + \ldots\ldots + \operatorname{Res} f(a_n)\right]$$

which is the desired result.

Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)} dx$, where C is the circle $|z| = 3$.

$$f(z) = \frac{\sin \pi^2 + \cos \pi z^2}{(z-1)^2 (z-2)}$$

Analytic within the circle $|z| = 3$ excepting the poles $z = 1$ and $z = 2$.

Since $z = 1$ is a pole of order 2.

$$\therefore \quad \operatorname{Res} f(1) = \frac{1}{1!}\left[\frac{d}{dz}\left\{(z-1)^2 f(z)\right\}\right]_{z=1} = \left[\frac{d}{dz}\left(\frac{\sin \pi^2 + \cos \pi z^2}{z-2}\right)\right]_{z=1}$$

$$= \left[\frac{(z-2)\left(2\pi z \cos \pi z^2 - 2\pi z \sin \pi z^2\right) - \left(\sin \pi z^2 + \cos \pi z^2\right)}{(z-2)^2}\right]_{z=1}$$

$$= (-1)(-2\pi) - (-1) = 2\pi + 1$$

Also $$\operatorname{Res} f(2) = \lim_{z \to 2}\left[(z-2) f(z)\right] = \lim_{z \to 2} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2} = 1$$

Hence by residue theorem,

$$\int_C f(z)\, dz = 2\pi i\left[\operatorname{Res} f(1) + \operatorname{Res} f(2)\right] = 2\pi i(2\pi + 1 + 1) = 4\pi(\pi + 1) i$$

Q. 5. (b) Expand $\frac{z^2 - 1}{(z+2)(z+3)}$ for $|z| = 3$

What is the difference between Taylor's series and Laurentz's series of a function?

Ans. Expand $\frac{z^2 - 1}{(z+1)(z+3)}$ for $|z| = 1$

By partial fraction

$$\frac{z^2 - 1}{(z+2)(z+3)} = \frac{z^2}{(z+2)(z+3)} - \frac{1}{(z+2)(z+3)}$$

$$= z^2(z+2)^{-1}(z+3)^{-1} - (z+2)^{-1}(z+3)^{-1} \quad ...(2)$$

For $|z|<1$ both $|z/2|$ and (z) are less than 1. Hence equation (2) is given as

$$f(z) = -\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+.....\right)$$

$$+\left(1+z+z^2+z^3+.....\right)$$

$$= \frac{1}{2}+\frac{3}{4}z+\frac{7}{8}z^2+\frac{15}{16}z^3+.....$$

Which is a Taylor's series.

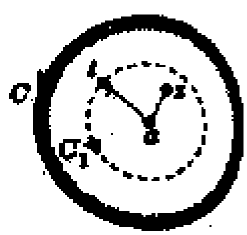
If f(z) is analytic inside a circle with C with centre at a, then for z inside C,

$$f(z+) = f(a)+f'(a)(z-a)+\frac{f''(a)}{2!}(z-a)^2+......+\frac{f^n(a)}{n!}(z-a)^n+...... \quad ...(i)$$

Proof : Let z be any point inside C. Draw a circle C_1 with centre at a enclosing z (Fig. 18.19). Let t be a point on C_1. We have

$$\frac{1}{t-z} = \frac{1}{t-a-(z-a)} = \frac{1}{t-a}\left(1-\frac{z-a}{t-a}\right)^{-1}$$

$$= \frac{1}{t-a}\left[1+\frac{z-a}{t-a}+\left(\frac{z-a}{t-a}\right)+.....+\left(\frac{z-a}{t-a}\right)^2+......\right] \quad ...(ii)$$



As $|z-a|<|t-a|$, i.e., $|(z-a)/(t-a)|<1$, this series converges uniformly. So, multiplying both sides of (ii) by f(t), we can integrate over C_1.

$$\therefore \quad \int_{C_1}\frac{f(t)}{t-z}dz = \int_{C_1}\frac{f(t)}{t-a}dt + (z-a)\int_{C_1}\frac{f(t)}{(t-a)^2}dt + \ldots\ldots + (z-a)^n\int_{C_1}\frac{f(t)}{(t-a)^{n+1}}dt + \ldots \qquad \ldots(iii)$$

Since f(t) is analytic on and inside C_1, therefore, applying the formulae (2) to (5) of p. 509 to (iii), we get (i) (i) which is known as Taylor's series.

Obs. Another remarkable fact is that complex analytic functions can always be represented by power series of the form (i).

(iii) Laurent's series :

If f(z) is analytic in the ring-shaped region R bounded by two concentric circles C and C_1 of radii *r* and $r_1(r > r_1)$ and with centre at a, then for all z in R

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots\ldots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \ldots$$

Where
$$a_n = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{(t-a)^{n+1}}dt,$$

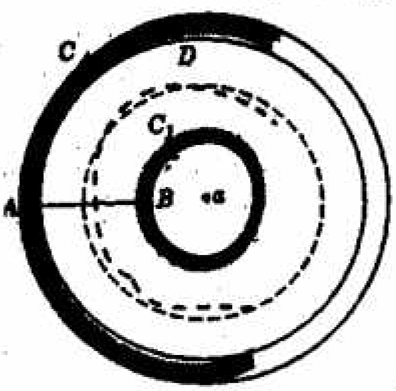
Γ being any curve in R, encircling C_1 (as in fig.)

Proof : Introduce cross-out AB, then f(z) is analytic in the region D bounded by AB, C_1 described clockwise, BA and C described anti-clockwise (see fig.). Then if z be any point in D, we have

$$f(z) = \frac{1}{2\pi i}\left[\int_{AB}\frac{f(t)}{t-z}dt + \int_{C_1}\frac{f(t)}{t-z}dt + \int_{BA}\frac{f(t)}{t-z}dt + \int_C\frac{f(t)}{t-z}dt\right]$$

$$= \frac{1}{2\pi i}\left[\int_C\frac{f(t)}{t-z}dt - \int_{C_1}\frac{f(t)}{t-z}dt\right] \qquad \ldots(i)$$

Where both C and C_1 are described anti-clockwise in (i) and integrals along AB and BA cancel.



For the first integral in (i), expanding $1/(t-z)$ as in & (2) we get

$$\frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}dt = \sum_{n=1}^{\infty}\frac{(z-a)^n}{2\pi i}\int_C \frac{f(t)}{(t-a)^{n+1}}dt$$

$$= \Sigma a_n (z-a)^n \text{ where } a_n = \frac{1}{2\pi i}\int_C \frac{f(t)}{(t-a)^{n+1}}dt$$

For the second integral in (i), let t lie on C_1. Then we write

Part-C

Q. 6. (a) There are three bags : first containing 1 white, 2 red, 3 green balls; second 2 white, 3 red, 1 green balls and third 3 white, 1 red, 2 green balls. Two balls are drawn from a bag chosen at random. They are found to be 1 red and 1 white. Find the probability that ball so drawn came from the second bag.

Ans. Bag 1

Number of white Ball = 1

Number of Red Ball = 2

Number of Green Balls = 3

Bag 2

Number of white Ball = 2

Number of Red Ball = 3

Number of Green Balls = 1

Bag 3

Number of white Ball = 3

Number of Red Ball = 1

Number of Green Balls = 2

Probability of two ball drawn from the bag

$$= \frac{2!}{6!4!}$$

$$= \frac{1}{720 \times 6}$$

Probability of 1 red ball $= \frac{2!}{1!} = 2$

Probability of 1 white ball = 1

Probability of getting ball from second bag

$$= \frac{2!}{6!2!}$$

$$= \frac{1}{6!}.$$

Q. 6. (b) Is the function defined as follows a density function?

$$f(x) = e^{-x},\ x \geq 0$$
$$= 0 \text{ otherwise}$$

If so find $P[1 \leq X \leq 2]$.

Ans. (i) Is the function defined as follows a density function

$$f(x) = e^{-x},\ x \geq 0$$
$$= 0,\ x < 0$$

(ii) If so, determine the probability that the variate having this density will fall in the interval (1, 2)?

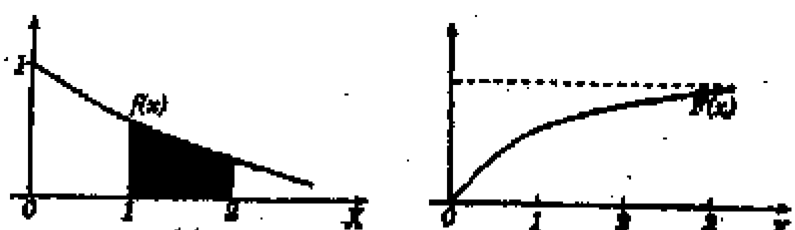
(iii) Also find the cumulative probability function F(2)?

(i) f(x) is clearly ≥ 0 for every x in (1, 2) and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0.dx + \int_{0}^{\infty} e^{-x}\,dx = 1$$

Hence the function f(x) satisfies the requirements for a density function.

(ii) Required probability = $P(1 \leq x \leq 2)$



(a) (b)

$$= \int_1^2 e^{-x}\,dx = e^{-1} - e^{-2} = 0.368 - 0.135 = 0.233$$

This probability is equal to the shaded area in fig.(a).

(iii) Cumulative probability function F(2)

$$\int_{-\infty}^{2} f(x)\,dx = \int_{-\infty}^{0} 0.dx + \int_0^2 e^{-x}\,dx$$

$$= 1 - e^{-2} = 1 - 0.135 = 0.865$$

Which is shown in fig. (b).

Q. 7. (a) Define Poisson distribution and discuss some of its properties.

Ans. 1. Poisson distribution :

It is a distribution related to the probabilities of events which are extremely rare, but which have a large number of independent opportunities for occurrence. The number of persons born blind per year in a large city and the number of deaths by horse kick in an army corps are some of the phenomena, in which this law is followed.

This distribution can be derived as a limiting case of the binomial distribution by makings very large and p very small, keeping np fixed (=m, say).

The probability of r successes in a binomial-distribution is

$$P(r) = {}^nC_r p^r q^{n-2} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} p^r q^{n-r}$$

$$= \frac{np(np-p)(np-2p)\ldots(np-r-1p)}{r!}(1-p)^{n-r}$$

As $n \to \infty$, $p \to 0$ $(np = m)$, we have

$$P(r) = \frac{m^r}{r!} \lim_{n\to\infty} \frac{(1-m/n)^n}{(1-m/n)^r} = \frac{m^r}{r!} e^{-m}$$

So that the probabilities of 0, 1, 2....., r, ... successes in a Poisson distribution are given by

$$e^{-m},\ me^{-m},\ \frac{m^2e^{-m}}{2!},\ldots\ldots,\frac{m^r e^{-m}}{r!},\ldots\ldots.$$

The sum of these probabilities in unity as it should be.

2. Constants of the Poisson distribution :

These constants can easily be derived from the corresponding constants of the binomial distribution simply by making $n \to \infty$, $p \to 0$, $(q \to 1)$ and noting that $np = m$

$$\text{Mean} = Lt\,(np) = m$$

$$\mu_2 = Lt(npq) = m\,Lt(q) = m$$

$\therefore$ Standard deviation $= \sqrt{m}$

Also $\mu_3 = m$, $\mu_4 = m + 3m^2$

$\therefore$ Skewness $\left(= \sqrt{\beta_1}\right) = 1/m$, Kurtosis $\left(= \beta_2\right) = 3 + 1/m$.

Since μ_3 is positive, Poisson distribution is positively skewed and since $\beta_2 > 3$, it is Leptoburitc.

(iii) Applications of Poisson distribution :

This distribution is applied to product concerning : (i) Arrival pattern of 'defective vehicles in a work-shop', patients in a hospital's 'telephone calls.'

(ii) Demand pattern for certain spare parts.

(iii) Number of fragments from a shell hitting a target.

(iv) Spatial distribution of bomb hits.

Q. 7. (b) Fit a normal curve to the following distribution

x :	2	4	6	8	10
f :	1	4	6	4	1

Ans.

x:	2	4	6	8	10
f:	1	4	6	4	1

$$\text{Mean} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{2+16+36+32+10}{16}$$

$$= \frac{96}{16}$$

$$= 6$$

∴ Mean of Poisson distribution

i.e., $m = 6$

Hence, the theoretical frequency for r successes is

$$\frac{Ne^{-m}(m)^r}{r!} = \frac{16e^{-6}(6)^r}{r!}$$

Where r = 0, 1, 2, 3, 4

∴ The theoretical frequencies are :

x:	2	4	6	8	10
f:	1	4	6	4	1

$$\left(\because e^{-6} = 0.61\right).$$

Q. 8. (a) Using Simplex method,

Maximize $z = 5x_1 + 3x_2$

subject to $x_1 + x_2 \le 2$,

$$5x_1 + 2x_2 \le 10$$

$$3x_1 + 8x_2 \le 12;\ x_1, x_2 \ge 0.$$

Ans.

Using simplex method

Minimize $Z = 5x_1 + 3x_2$

Subject to $x_1 + x_2 \le 2$, $5x_1 + 2x_2 \le 10$, $3x_1 + 8x_2 \le 12$, $x_1 x_2 \ge 0$

Solution consists of the following steps :

Step 1 : Check whether the objective function is to be maximized and all b's are positive. The problem being of maximization type and all b's being ≥ 0, this step is not necessary.

Step 2 : Express the problem in the standard form.

By introducing the slack variables s_1, s_2, s_3, the problem in standard form becomes

$$\text{Max. } Z = 5x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3$$

$$\text{Subject to } x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 2$$

$$5x_1 + 2x_2 + 0s_1 + s_2 + 0s_3 = 10$$

$$3x_1 + 8x_2 + 0s_1 + 0s_2 + 0s_3 = 10$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

Step 3 : Find an initial basic feasible solution

There are three equations involving five unknowns and for obtaining a solution, we assign zero values to any two of the variables. We start with a basic solution for which we set $x_1 = 0$ and $x_2 = 0$. (This basic solution corresponds to the origin in the graphical method). Substituting $x_1 = x_2 = 0$ in (i), (ii) and (iii), we get the basic solution

$$s_1 = 2, s_2 = 10, s_3 = 12$$

Since all s_1, s_2, s_3 are positive, the basic solution is also feasible and non-degenerate.

$\therefore$ The basic feasible solution is given by the following table :

	c_j	5	3	0	0	0		
c_B Basic x_1		x_2	s_1	s_2	s_3	b	0	
0	s_1	(1)	1	1	0	0	2	2/1 ←
0	s_2	5	2	0	1	0	10	10/5
0	s_3	3	8	0	0	1	12	12/3
	$Z_j = \Sigma c_B a_{ij}$	0	0	0	0	0	0	
	$C_j = c_j - Z_j$	5	3	0	0	0		
		↑						

[For x_1 – column $(j = 1)$, $Z_j = \Sigma c_B a_{i1} = 0(1) + 0(5) + 0(3) = 0$

And for x_2 – column $(j = 2)$, $Z_j = \Sigma c_B a_{i2} = 0(1) + 0(2) + 0(8) = 0$.

Similarly $\quad Z_j = 0(2) + 0(10) + 0(12) = 0.1$

Step 4 : Apply optimality test.

As C_j is positive under some columns, the initial basic feasible solution is not optimal (i.e., can be improved) and we proceed to the next step.

Step 5 : (i) Identify the incoming and outgoing variables.

The above table shows that x_1 is the incoming variable as its incremental contributions $C_j (= 5)$ is maximum and the column in which it appears is the key column (shown marked by an arrow at the bottom).

Dividing the elements under b-column by the corresponding elements of key-column, we find minimum positive ratio θ is 2 in two rows. We, therefore, arbitrarily choose the row containing s_1 as the key row (shown marked by an arrow on its right end). The element at the intersection of key row an the key column i.e., (1), the key element. s_1 is therefore, the outgoing basic variable which will now become non-basic.

Having decided that x_1 is to enter the solution, we have tried to find as to what maximum value x_1 could have without violating the constraints. So, removing s_1, the new basic will contains x_1, s_2 and s_3 as the basic variables.

(ii) Iterate towards the optimal solution :

To transform the initial set of equations with a basic feasible solution into an equivalent set equations with a different basic feasible solution, we make the key element unity. Here the by element being unity, we retain the key row as it is. Then to make all other elements in key column zero, we subtract proper multiples of key row from the other rows. Here we subtract 5 uses the elements of key row from the second row and 3 times the elements of key row from the ... row. These become the second and the third rows of the next table. We also change the corresponding value under c_B column from 0 to 5, while replacing s_1 by x_1 under the basis. Thus, be second basic feasible solution is given by the following table :

		c_j	5	3	0	0	0	
c_B	Basic	x_1	x_2	s_1	s_2	s_3	b	θ
5		x_1	1	1	1	0	0	2
0		s_2	0	–3	–5	1	0	0
0		s_3	0	5	–3	0	1	6
		$Z_j = \Sigma c_B a_{ij}$	5	5	5	0	0	10
		$C_j = c_j - Z_j$	0	–2	–5	0	0	

As C_j is either zero or negative under all columns, the above table gives the optimal basic table solution. This optimal solution is $x_1 = 2$, $x_2 = 0$ and maximum Z = 10.

Q. 8. (b) Using dual Simplex method

Maximize $z_r = -3x_1 - x_2$

Subject to $x_1 + x_2 \geq 1,$

$2x_1 + 3x_2 \geq 2;$

$x_1, x_2 \geq 0.$

Ans. Consists of the following steps :

Step 1 : (i) Convert the first and third constraints into $(\leq)$ type. These constraints become

$$-x_1 - x_2 \leq -1, \; -x_1 - 2x_2 \leq -10$$

(ii) Express the problem in standard form

Introducing slack variables s_1, s_2, s_3, s_4 the given problem takes the form

$$\text{Max. } Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$$

Subject to $-x_1 - x_2 + s_1 = -1, \; x_1 + x_2 + s_2 = 7, \; -x_1 - 2x_2 + s_3 = -10,$

$x_2 + s_4 = 3, \; x_1, x_2, s_1, s_2, s_3, s_4 \geq 10$

Step 2 : Find the initial basic solution

Setting the decision variables x_1, x_2 each equal to zero, we get the basic solution

$$x_1 = x_2 = 0; \; s_1 = -1, \; s_2 = 7, \; s_3 = -10, \; s_4 = 3 \text{ and } Z = 0$$

$\therefore$ Initial solution is given by the table below :

	c_j	–3	–2	0	0	0	0	
c_B	Basic x_1	x_2	s_1	s_2	s_3	s_4	b	
0	s_1	–1	–1	1	0	0	0	–1
0	s_2	1	1	0	1	0	0	7
0	s_3	–1	(–2)	0	0	1	0	–10 ←
	$Z_j = \Sigma c_B a_{ij}$	0	0	0	0	0	0	0
	$C_j = c_j - Z_j$	–3	–2	0	0	0	0	
			↑					

Step 3 : Test nature of C_j.

Since all C_j values are ≤ 0 and $b_1 = -1$, $b_3 = -10$, the initial solution is optimal but infeasible. We therefore, proceed further.

Step 4 : Mark the outgoing variable.

Since b_3 is negative and numerically largest, the third row is the key row and s_3 is the outgoing variable.

Step 5 : Calculate ratios of elements in C_j-row to the corresponding negative elements of the key row.

These ratios are $-3/-1=3$, $-2/-2 = 1$ (negative ratios corresponding to +ve or zero elements of key row). Since the smaller ratio is 1, therefore, x_2-column is the key column and (-2) is the key element.

Step 6 : Iterate towards optimal feasible solution.

(i) Drop s_3 and introduce x_2 alongwith its associated value -2 under c_B column. Convert the key element to unity and make all other elements of the key column zero. Then the second solution is given by the table below :

		c_j	-3	-2	0	0	0	0	
c_B	Basic	x_1	x_2	s_1	s_2	s_3	s_4	b	
0		s_1	$-\frac{1}{2}$	0	1	0	$-\frac{1}{2}$	0	4
0		s_2	$\frac{1}{2}$	0	0	1	$\frac{1}{2}$	0	2
2		x_2	$\frac{1}{2}$	1	0	0	$-\frac{1}{2}$	0	5
0		s_4	$\left(-\frac{1}{2}\right)$	0	0	0	$\frac{1}{2}$	1	$-2 \leftarrow$
	$Z_j = \Sigma c_B a_{ij}$		-1	-2	0	0	1	0	-10
	$C_j = c_j - Z_j$		-2	0	0	0	-1	0	
				$\uparrow$					

Since all C_j values are ≤ 0 and $b_4 = -2$, this solution is optimal but infeasible. We theref
further.

(ii) Mark the outgoing variable.

Since b_4 is negative, the fourth row is the key row and s_4 is the outgoing variable.

(iii) Calculate ratios of elements in C_j -row to the corresponding negative elements of the key

This ratios is $-2/-\frac{1}{2} = 4$ (neglecting other ratios corresponding to +ve or 0 elements of key r

$\therefore$ x_1 -column is the key column and $\left(-\frac{1}{2}\right)$ is the key element.

(iv) Drop s_4 and introduce x_1 with its associated value -3 under the c_B column. Convert the ele
ty and make all other elements of the key column zero. Then the third solution given by the table be

	c_j	-3	-2	0	0	0	0	
c_B	Basic	x_1	x_2	s_1	s_2	s_3	s_4	b
0	s_1	0	0	1	0	-1	-1	6
0	s_2	0	0	0	1	1	1	0
2	x_2	0	1	0	0	0	1	3
3	x_1	1	0	0	0	-10	-2	4
	Z_j	-3	-2	0	0	3	4	-18