B.B.2nd Sem. Examination May-2009
Paper : Math-102 - E

Note:- Attempt **five** questions in all, selecting at least **one** question from each part.

PART-A

1. (a) For the Matrix $A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ **find non-singular matrices** P **and** Q **such that PAQ is in the normal form. Hence find the rank of A.**

Solution. We write $A_{3\times 4} = I_3 A I_4$

$$\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

operating $R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

operating $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

operating $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, $C_4 \to C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

operating $R_3 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -5 & -10 \\ 0 & -6 & -2 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -2 \\ 1 & 0 & -3 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

operating $R_3 \to R_3 - 6R_2$, $R_2 \to -R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 10 \\ 0 & 0 & 28 & 56 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ -6 & 1 & 9 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

operating $C_4 \to C_4 - 2C_3$, $C_3 \to C_3 - 5C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 28 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ -6 & 1 & 9 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

operating $R_4 \to \frac{1}{28} R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 2 \\ \frac{-3}{14} & \frac{1}{28} & \frac{9}{28} \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad [I_3 : 0] = PAQ \qquad \ldots(1)$$

Where $$P = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 2 \\ \frac{-3}{14} & \frac{1}{28} & \frac{9}{28} \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From equation (1) normal form of given matrix A is $[I_3 \ : \ 0]$.
$\therefore$ Rank of Matrix A is 3. **Ans.**

1. (b) Discuss the consistency of the system of equations :

$$\mathbf{2x - 3y + 6z - 5w = 3}$$
$$\mathbf{y - 4z + w = 1}$$
$$\mathbf{4x - 5y + 8z - 9w = \lambda}$$

for various values of λ, if consistent, find the solution.

Solution. $2x - 3y + 6z - 5w = 3$

$$y - 4z + w = 1$$
$$4x - 5y + 8z - 9w = \lambda$$

$\therefore$ Augumented matrix $= \begin{bmatrix} 2 & -3 & 6 & -5 & : & 3 \\ 0 & 1 & -4 & 1 & : & 1 \\ 4 & -5 & 8 & -9 & : & \lambda \end{bmatrix}$

operating $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 2 & -3 & 6 & -5 & : & 3 \\ 0 & 1 & -4 & 1 & : & 1 \\ 0 & 1 & -4 & 1 & : & \lambda - 6 \end{bmatrix}$$

operating $R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 2 & -3 & 6 & -5 & : & 3 \\ 0 & 1 & -4 & 1 & : & 1 \\ 0 & 0 & 0 & 0 & : & \lambda - 7 \end{bmatrix} \qquad ...(1)$$

(i) if $\lambda = 7$ then $\rho(A) = 2$, $\rho(A : B) = 2$

$\because$ $\rho(A) = \rho(A:B) = 2 <$ number of unknowns which is 3

$\therefore$ systems of equations has infinite number of solutions.

(ii) If $\lambda \neq 7$ then $\rho(A) = 2$, $\rho(A:B) = 3$

$\because$ $\rho(A) \neq \rho(A:B)$

$\therefore$ systems of equations has no solution.

If $\lambda = 7$, from (1)

$$\sim \begin{bmatrix} 2 & -3 & 6 & -5 & : 3 \\ 0 & 1 & -4 & 1 & : 1 \\ 0 & 0 & 0 & 0 & : 0 \end{bmatrix}$$

or $\quad 2x - 3y + 6z - 5w = 3 \qquad ...(2)$

$\quad y - 4z + w = 1 \qquad ...(3)$

let $\quad z = k_1$ and $w = k_2$

from (2) $\quad y - 4k_1 + k_2 = 1$

or $\quad y = 1 + 4k_1 - k_2$

from (2) $\quad 2x - 3(1 + 4k_1 - k_2) + 6k_1 - 5k_2 = 3$

$\quad 2x - 3 - 12k_1 + 3k_2 + 6k_1 - 5k_2 = 3$

$\quad 2x - 3 - 6k_1 - 2k_2 = 3$

or $\quad 2x = 6 + 6k_1 + 2k_2$

or $\quad x = 3 + 3k_1 + k_2$

$\therefore$ $\quad x = 3 + 3k_1 + k_2$

$$y = k_2$$

$$z = k_1 \quad \text{(where } k_1 \text{ and } k_2 \text{ are arbitrary).} \quad \textbf{Ans.}$$

2. (a) (i) Prove that the sum of the eigen values of a matrix A is the sum of the elements of the principal diagonal.

Proof : We prove this property for a matrix of order 3. However, this result is true for a matrix of any order.

Let
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$\therefore$ Characteristic equation is
$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix}$$

$$= -\lambda^3 + \lambda^2(a_{11} + a_{22} + a_{33}) \ldots\ldots\ldots \qquad \ldots(1)$$

If $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A, then we get

$$|A - \lambda I| = (-1)^3(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$

$$= -\lambda^3 + \lambda^2(\lambda_1 + \lambda_2 + \lambda_3) - \ldots\ldots\ldots \qquad \ldots(2)$$

From (1) and (2), we get

$$-\lambda^3 + \lambda^2(a_{11} + a_{22} + a_{33}) \ldots\ldots = -\lambda^3 + \lambda^2(\lambda_1 + \lambda_2 + \lambda_3) \ldots\ldots$$

Comparing Co-efficient of λ^2 on both sides, we get

$$\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$$

= Sum of the elements of the principal diagonal.

2. (a) (ii) Find the sum and product of the eigen values of the matrix :

$$\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{2} & \mathbf{1} & \mathbf{5} & \mathbf{6} \\ \mathbf{7} & \mathbf{4} & \mathbf{3} & \mathbf{2} \\ \mathbf{4} & \mathbf{3} & \mathbf{0} & \mathbf{5} \end{bmatrix}$$

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 5 & 6 \\ 7 & 4 & 3 & 2 \\ 4 & 3 & 0 & 5 \end{bmatrix}$$

$\therefore$ Sum of Eigen values = 1 + 1 + 3 + 5 = 10. **Ans.**

and product of Eigen values = value of $|A|$

$$= \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 5 & 6 \\ 7 & 4 & 3 & 2 \\ 4 & 3 & 0 & 5 \end{vmatrix} = 262 \quad \textbf{Ans.}$$

2. (b) Verify that the following set of vectors in R^3 is linearly depedent : (1, 0, 1) , (1, 1, 1), (1, 1, 2) and (1, 2, 1) . Also find the number of linearly independent vectors.

Solution. Let $x_1 = (1,0,1)$, $x_2 = (1,1,1)$, $x_3 = (1,1,2)$ and $x_4 = (1, 2, 1)$

Adding suitable multiples of x_1 to x_2 , x_3 and x_4 so that the first component reduces to zero, we have

$$x_1 - x_2 = (0, -1, 0) \qquad ...(1)$$

$$x_1 - x_3 = (0, -1, -1) \qquad ...(2)$$

$$x_1 - x_4 = (0, -2, 0) \qquad ...(3)$$

From (1) and (2), we get

$$x_1 - x_2 - x_1 + x_3 = (0, 0, 1)$$

or $$x_3 - x_2 = (0, 0, 1) \qquad ...(4)$$

From (2) and (3) $$x_1 - x_4 - 2x_1 + 2x_3 = (0, 0., 2)$$

or $$2x_3 - x_1 - x_4 = (0, 0, 2) \qquad ...(5)$$

To reduced to 3rd component to zero, multiplying equation (4) by 2 and subtracting from equation (5), we get

$$2x_3 - x_1 - x_4 - 2(x_3 - x_2) = (0, 0, 0)$$

or $$x_1 - 2x_2 + x_4 = (0, 0, 0)$$

Thus, their exist numbers $k_1 = 1, k_2 = -2, k_3 = 0, k_4 = 1$
which are not all zero such that

$$k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 = O \qquad ...(6)$$

Hence the vectors x_1 , x_2 and x_3 are linearly dependent and equation (6) is the relation between them.

3. (a) Solve $(x^2y^2 + xy + 1)y\,dx + (x^2y^2 - xy + 1)x\,dy = 0$

Solution. The given equation is

$$(x^2y^2 + xy + 1)y\,dx + (x^2y^2 - xy + 1)x\,dy = 0 \qquad ...(1)$$

Comparing it with $Mx + Ndy = 0$, we get

$$M = (x^2y^2 + xy + 1)y = x^2y^3 + xy^2 + y$$

and $$N = (x^2y^2 - xy + 1)x = x^3y^2 - x^2y + x$$

$\therefore$ $$\frac{\partial M}{\partial y} = 3x^2y^2 + 2xy + 1 \text{ and } \frac{\partial N}{\partial x} = 3x^2y^2 - 2xy + 1$$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, thus equation (1) is not exact.

also
$$\begin{aligned} Mx - Ny &= (x^2y^3 + xy^2 + y)x - (x^3y^2 - x^2y + x)y \\ &= x^3y^3 + x^2y^2 + xy - x^3y^3 + x^2y^2 - xy \\ &= 2x^2y^2 \neq 0 \end{aligned}$$

Equation (1) is of the form $f_1(xy)y\,dx + f_2(xy)x\,dy = 0$

$$\therefore \qquad I.F. = \frac{1}{Mx - Ny} = \frac{1}{2x^2y^2}$$

Multiplying (1) by $\frac{1}{2x^2y^2}$, we get

$$\left[\frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y}\right]dx + \left[\frac{x}{2} - \frac{1}{2y} + \frac{1}{2xy^2}\right]dy = 0 \qquad \ldots(2)$$

Here $\qquad M = \frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y}$ and $N = \frac{x}{2} - \frac{1}{2y} + \frac{1}{2xy^2}$

$$\therefore \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{1}{2} - \frac{1}{2x^2y^2}$$

Therefore, equation (2) is an exact equation.

Hence the solution is $\displaystyle\int_{y\,\text{constant}} \left[\frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y}\right]dx - \int \frac{1}{2y}\,dy = c_1$

$$\frac{1}{2}xy + \frac{1}{2}\log x - \frac{1}{2xy} - \frac{1}{2}\log y = c_1 \quad \text{or} \quad xy - \frac{1}{xy} + \log\left(\frac{x}{y}\right) = c\text{. } \textbf{Ans.}$$

3. (b) Solve for the current $I(t)$ in an RL circuit if $R = 2$ ohms, $L = 25$ henrys and $E(t) = Ae^{-t}$ with $A > 0$, as a constant and $I(0) = 0$.

Solution. Let i be the current in the circuit at any time t, then by Kirchoff's law, we get

$$L\frac{di}{dt} + Ri = E$$

But $L = 25$ hanrys, $R = 2$ ohms and $E = Ae^{-t}$

$$\therefore \qquad 25\frac{di}{dt} + 2i = Ae^{-t} \quad \text{or} \quad \frac{di}{dt} + \frac{2}{25}i = \frac{A}{25}e^{-t} \qquad \ldots(1)$$

$$\therefore \qquad I.F. = e^{\int \frac{2}{25}dt} = e^{\frac{2t}{25}}$$

Hence solution of equation (1) is

$$i \cdot e^{\frac{2t}{25}} = \int \frac{A}{25}e^{-t} \cdot e^{\frac{2t}{25}}\,dt + k = \frac{A}{25}\int e^{-23t/25}\,dt + k$$

$$= \frac{A}{25} \times \frac{e^{-23t/25}}{\frac{-23}{25}} + k \qquad = \frac{-A}{23}e^{-23t/25} + k$$

$$\therefore \qquad i = \frac{-A}{23}e^{-t} + k \qquad \ldots(2)$$

Put $i = 0$, $t = 0$

$$\therefore \qquad 0 = \frac{-A}{23}e^0 + k \Rightarrow k = \frac{A}{23}$$

From (2), we get

$$i = \frac{-A}{23}e^{-t} + \frac{A}{23} = \frac{A}{23}[1-e^{-t}]. \textbf{ Ans.}$$

4. (a) Solve $(D^2-4)y = x\sinh x$

Solution. Given equation is $(D^2-4)y = x\sin h\, x$

Its A.E. is $\quad D^2-4 = 0$

$$(D+2)(D-2) = 0$$

$$\therefore \qquad D = 2, -2$$

$$\therefore \qquad C.F. = c_1e^{2x} + c_2e^{-2x}$$

And

$$P.I. = \frac{1}{(D^2-4)}(x\sin h\, x)$$

$$= \frac{1}{(D^2-4)}x\left[\frac{e^x-e^{-x}}{2}\right] \qquad \left[\because \sin hx = \frac{e^x-e^{-x}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(D^2-4)}e^x\cdot x - \frac{1}{(D^2-4)}e^{-x}\cdot x\right]$$

$$= \frac{1}{2}\left[e^x\cdot\frac{1}{\left[(D+1)^2-4\right]}x - \frac{e^{-x}}{\left[(D-1)^2-4\right]}\cdot x\right]$$

$$= \frac{1}{2}\left[e^x\cdot\frac{1}{\left(D^2+2D+1-4\right)}\cdot x - e^{-x}\frac{1}{\left(D^2-2D+1-4\right)}\cdot x\right]$$

$$= \frac{1}{2}\left[e^x\cdot\frac{1}{(D^2+2D-3)}\cdot x - e^{-x}\frac{1}{(D^2-2D-3)}\cdot x\right]$$

$$= \frac{1}{2}\left[e^x\cdot\frac{1}{-3\left(1-\frac{2D}{3}-\frac{D^2}{3}\right)}\cdot x - e^{-x}\cdot\frac{1}{-3\left(1+\frac{2D}{3}-\frac{D^2}{3}\right)}\cdot x\right]$$

$$= -\frac{1}{6}\left[e^x\left\{1-\left(\frac{2D}{3}+\frac{D^2}{3}\right)\right\}^{-1}x - e^{-x}\left\{1+\left(\frac{2D}{3}-\frac{D^2}{3}\right)\right\}^{-1}x\right]$$

$$= -\frac{1}{6}\left[e^{x}\left(1+\frac{2D}{3}+\ldots\ldots\right)x - e^{-x}\left(1-\frac{2D}{3}+\ldots\ldots\right)x\right]$$

$$= -\frac{1}{6}\left[e^{x}\left\{x+\frac{2}{3}D(x)\right\} - e^{-x}\left\{x-\frac{2}{3}D(x)\right\}\right]$$

$$= -\frac{1}{6}\left[e^{x}\left(x+\frac{2}{3}\right) - e^{-x}\left(x-\frac{2}{3}\right)\right]$$

$$= -\frac{1}{6}\left[e^{x}\cdot x+\frac{2}{3}e^{x} - xe^{-x}+\frac{2}{3}e^{-x}\right]$$

$$= -\frac{1}{6}\left[x(e^{x}-e^{-x})+\frac{2}{3}(e^{x}+e^{-x})\right]$$

$$= -\frac{x}{3}\left[\frac{e^{x}-e^{-x}}{2}\right]-\frac{2}{9}\left[\frac{e^{x}+e^{-x}}{2}\right]$$

$$= -\frac{x}{3}\sin h\,x-\frac{2}{9}\cos h\,x \qquad \left[\because \sin hx = \frac{e^{x}-e^{-x}}{2},\ \cos hx = \frac{e^{x}+e^{-x}}{2}\right]$$

Hence C.S. is $\quad y = C.F. + P.I.$

$$y = c_1e^{2x}+c_2e^{-2x}-\frac{x}{3}\sin h\,x-\frac{2}{9}\cos h\,x\text{. } \textbf{Ans.}$$

4. (b) Solve the differential equation : $(x^2D^2 - xD - 3)y = x^2(\log x)^2$

Solution. $\quad (x^2D^2 - xD - 3)y = x^2(\log x)^2$

This equation is a Cauchy's homogeneous linear equation.
Put $x = e^z$ *i.e.* $z = \log x$

so that $xD = Dy$, $x^2D^2 = D(D-Dy)$, where $D = \frac{d}{dz}$

These value putting in given equation, we get

$$[D(D-1)-D-3]y = e^{2z}z^2$$

or $\quad (D^2-2D-3)y = z^2e^{2z}$

Its *A.E.* is $D^2-2D-3 = 0$

or $\quad D = 3, -1$

$\therefore \quad C.F. = c_1e^{3z}+c_2e^{-z} = c_1x^3+\frac{c_2}{x}$

$$P.I. = \frac{1}{D^2-2D-3}(e^{2z}\cdot z^2)$$

$$= \frac{e^{2z}}{[(D+2)^2 - 2(D+2) - 3]} \cdot z^2$$

$$= e^{2z} \cdot \frac{1}{D^2 + 2D - 3} \cdot z^2$$

$$= e^{2z} \cdot \frac{1}{-3\left[1 - \frac{2D}{3} - \frac{D^2}{3}\right]} z^2 = \frac{1}{3} e^{2z} \left[1 - \left(\frac{2D}{3} + \frac{D^2}{3}\right)\right]^{-1} z^2$$

$$= -\frac{1}{3} e^{2z} \left[1 + \left(\frac{2D}{3} + \frac{D^2}{3}\right) + \left(\frac{2D}{3} + \frac{D^2}{3}\right)^2 + \ldots\ldots\right] z^2$$

$$= -\frac{1}{3} e^{2z} \left[1 + \frac{2D}{3} + \frac{D^2}{3} + \frac{4D^2}{9}\right] z^2 = -\frac{1}{3} e^{2z} \left[z^2 + \frac{4z}{3} + \frac{14}{9}\right]$$

$$= \frac{-x^2}{3} \left[(\log x)^2 + \frac{4}{3}(\log x) + \frac{14}{9}\right]$$

Hence C.S. in $y = c_1 x^3 + \frac{c_2}{x} - \frac{x^2}{3}\left[(\log x)^2 + \frac{4}{3}(\log x) + \frac{14}{9}\right]$. **Ans.**

5. (a) Solve the differential equation by the method of variations of parameters :

$$y'' + y = \sec^2 x$$

Solution. $y'' + y = \sec^2 x$ or $(D^2 + 1)y = \sec^2 x$

Its *A.E.* is $D^2 + 1 = 0 \Rightarrow D = \pm i$

$\therefore$ *C.F.* is $y = c_1 \cos x + c_2 \sin x$

Here $y_1 = \cos x$, $y_2 = \sin x$ and $x = \sec^2 x$

$\therefore$ $W = \begin{vmatrix} y_1 & y_2 \\ y_1^1 & y_2^1 \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

and $P.I. = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X\, dx}{W}$

$$= -\cos x \int \sin x \sec^2 dx + \sin x \int \cos x \cdot \sec^2 x\, dx$$

$$= -\cos x \int \tan x \sec x\, dx + \sin x \int \cos x\, dx$$

$$= -\cos x \sec x + \sin x \cdot \sin x$$

$$= -1 + \sin^2 x$$

$$= -1 + 1 - \cos^2 x = -\cos^2 x$$

Here $y = C.F. + P.I.$

$$y = c_1 \cos x + c_2 \sin x - \cos^2 x \text{. } \textbf{Ans.}$$

5. (b) Determine the current $I(t)$ in an LCR circuit with $e.m.f.$ $E(t) = E_0 \sin \omega t$ in case the circuit is tuned to resonance so that $\omega^2 = 1/LC$ and R/L is so small that second and higher degree terms can be rejected. Assuming that at $t = 0$, $I(0) = I'(0) = 0$.

Solution. The differential equation is $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{C} = E \sin \omega t$

or $$\left(LD^2 + RD + \frac{1}{C}\right)q = E \sin \omega t \text{, where } D = \frac{d}{dt} \qquad \dots(1)$$

Its $A.E.$ is $LD^2 + RD + \dfrac{1}{C} = 0$, so that

$$D = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L} = \frac{-R \pm i\sqrt{\frac{4L}{C} - R^2}}{2L} \text{, since } R^2 < \frac{4L}{C}$$

$$= \frac{R}{2L} \pm i\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = -\frac{R}{2L} \pm ip \text{, } \left[\because \frac{1}{LC} - \frac{R^2}{4L^2} = p^2\right]$$

Its $$C.F. = e^{-\frac{Rt}{2L}}(c_1 \cos pt + c_2 \sin pt)$$

and $$P.I. = \frac{1}{LD^2 + RD + \frac{1}{C}} E \sin \omega t$$

$$= E \cdot \frac{1}{-L\omega^2 + RD + \frac{1}{C}} \sin \omega t = \frac{E}{R} \cdot \frac{1}{D} \sin \omega t \text{, since } \omega^2 = \frac{1}{LC}$$

$$= -\frac{E}{R\omega} \cos \omega t$$

$\therefore$ The complete solution of equation (1) is $q = e^{-\frac{Rt}{2L}}(c_1 \cos pt + c_2 \sin pt) - \dfrac{E}{R\omega} \cos \omega t \qquad \dots(2)$

Initially, when t = 0, q = 0 $\qquad \therefore \qquad c_1 = \dfrac{E}{R\omega}$

Differentiating (2) w.r.t. t

$$\frac{dq}{dt} = e^{-\frac{Rt}{2L}}(-pc_1 \sin pt + pc_2 \cos pt) - \frac{R}{2L} e^{-\frac{Rt}{2L}}(c_1 \cos pt + c_2 \sin pt) + \frac{E}{R} \sin \omega t$$

Initially, when $t = 0$, $\dfrac{dq}{dt} = i = 0$

$$\therefore \qquad pc_2 - \frac{R}{2L}c_1 = 0 \qquad \text{or} \qquad c_2 = \frac{R}{2pL} \cdot \frac{E}{RW} = \frac{E}{2pL\omega}$$

Substituting the values of c_1 and c_2 in equation (2), we get

$$q = e^{-\frac{Rt}{2L}}\left(\frac{E}{R\omega}\cos pt + \frac{E}{2pL\omega}\sin pt\right) - \frac{E}{R\omega}\cos\omega t$$

or
$$q = \frac{E}{R\omega}\left[-\cos\omega t + e^{-\frac{Rt}{2L}}\left(\cos pt + \frac{R}{2Lp}\sin pt\right)\right] \quad \ldots (3)$$

Differentiating (3) w.r.t. t

$$\frac{dq}{dt} = \frac{E}{R\omega}\left[\omega\sin\omega t - \frac{R}{2L}e^{-\frac{Rt}{2L}}\left(\cos pt + \frac{R}{2Lp}\sin pt\right) + e^{-\frac{Rt}{2L}}\left(-p\sin pt + \frac{R}{2L}\cos pt\right)\right]$$

$$\therefore \quad i = \frac{E}{R\omega}\left[\omega\sin\omega t - e^{-\frac{Rt}{2L}}\left(\frac{R^2}{4L^2p} + p\right)\sin pt\right] = \frac{E}{R\omega}\left[\omega\sin\omega t - e^{-\frac{Rt}{2L}} \cdot \frac{R^2 + 4L^2p^2}{4L^2p}\sin pt\right]$$

$$= \frac{E}{R}\left[\sin\omega t - e^{-\frac{Rt}{2L}} \cdot \frac{1}{LCp\omega}\sin pt\right] \qquad \left[\because \frac{R^2 + 4L^2p^2}{4L^2} = \frac{1}{LC}\right]$$

$$= \frac{E}{R}\left[\sin\omega t - e^{-\frac{Rt}{2L}} \cdot \frac{\sqrt{LC}}{LCp}\sin pt\right] \qquad \left[\because \omega^2 = \frac{1}{LC}\right]$$

or
$$i = \frac{E}{R}\left[\sin\omega t - \frac{1}{p\sqrt{LC}}e^{-\frac{Rt}{2L}}\sin pt\right]$$

6. (a) Find Laplace transform of each of the following :

(i) $e^{4t}\sin 2t\cos t$ (ii) $\sin h(t)\cos^2 t$ (iii) $e^{-t}\cos^2 t$.

Solution. (a) (i) We have $L[\sin 2t\cos t] = \frac{1}{2}L[2\sin 2t\cos t]$

$$= \frac{1}{2}L[\sin(2t+t) + \sin(2t-t)]$$

$$= \frac{1}{2}L[\sin 3t + \sin t]$$

$$= \frac{1}{2}[L(\sin 3t) + L(\sin t)]$$

$$= \frac{1}{2}\left[\frac{3}{s^2+3^2} + \frac{1}{s^2+1^2}\right]$$

$\therefore$ By first shifting property, we get

$$L\left[e^{4t}\sin 2t\cos t\right] = \frac{1}{2}\left[\frac{3}{(s-4)^2+9} + \frac{1}{(s-4)^2+1}\right]$$

$$= \frac{1}{2}\left[\frac{3}{s^2-8s+25} + \frac{1}{s^2-8s+17}\right] \textbf{ Ans.}$$

Solution. 6. (a) (ii) $L\left[\sinh t\cos^2 t\right]$

$$= L\left[\left(\frac{e^t - e^{-t}}{2}\right)\left(\frac{1+\cos 2t}{2}\right)\right]$$

$$= \frac{1}{4}L\left[e^t - e^{-t} + e^t\cos 2t - e^{-t}\cos 2t\right]$$

$$= \frac{1}{4}L\left[\frac{1}{s-1} - \frac{1}{s+1} + \frac{s-1}{(s-1)^2+4} - \frac{s+1}{(s+1)^2+4}\right]$$

$$= \frac{1}{4}\left[\frac{2}{s^2-1} + \frac{s-1}{s^2-2s+5} - \frac{s+1}{s^2+2s+5}\right].$$ **Ans.**

Solution. 6. (a) (iii) $L\left[e^{-t}\cos^2 t\right]$

$$L\left[e^{-t}\left(\frac{1+\cos 2t}{2}\right)\right]$$

$$\frac{1}{2}L\left[e^{-t} + e^{-t}\cos 2t\right]$$

$$\frac{1}{2}\left[\frac{1}{s+1} + \frac{s+1}{(s+1)^2+4}\right]$$ **Ans.**

6. (b) Solve the simultaneous differenial equations using Laplace transforms :

$$x'(t) + y'(t) + x(t) = -e^{-t}, x'(t) + 2y'(t) + 2x(t) + 2y(t) = 0, \text{ where } x(0) = -1,\ y(0) = 1.$$

Solution. $x'(t) + y'(t) + x(t) = -e^{-t}$

$$x'(t) + 2y'(t) + 2x(t) + 2y(t) = 0$$

Taking Laplace of eq. (1), we have

$$L[x'] + L[y'] + L[x] = -L\left[e^{-t}\right]$$

$$s\bar{x} - x(0) + s\bar{y} - y(0) + \bar{x} = -\frac{1}{s+1}$$

$$s\bar{x} + 1 + s\bar{y} - 1 + \bar{x} = \frac{-1}{s+1} \qquad [\because\ x(0) = -1,\ y(0) = 1]$$

$$s\bar{x}(s+1) + s\bar{y} = -\frac{1}{s+1} \qquad \ldots(1)$$

Taking Laplace of eq. (2), we have

$$s\bar{x} - x(0) + 2s\bar{y} - 2y(0) + 2\bar{x} + 2\bar{y} = 0$$

$$s\bar{x} + 1 + 2s\bar{y} - 2 + 2\bar{x} + 2\bar{y} = 0 \qquad [\because\ x(0) = -1,\ y(0) = 1]$$

$$\bar{x}(s+2) + (2s+2)\bar{y} = 1 \qquad \ldots(2)$$

Multiplying eq. (1) by (2s+2) and (2) by s and subtracting (1) from (2),

$$\bar{x} = \frac{-(s^2+3s+2)}{(s+1)(s^2+2s+2)}$$

Taking Laplace inverse on both sides, we get

$$x = -L^{-1}\left[\frac{s}{s^2+2s+2}+\frac{2}{s^2+2s+2}\right]$$

$$x = -e^{-t}\left[\cos t+2\sin t\right]$$

Similarly $$\bar{y} = \frac{s^2+3s+3}{(s+1)(s^2+2s+2)}$$

Taking Laplace inverse on both sides, we get

$$y = e^{-t}\left[\cos t+2\sin t\right]. \textbf{ Ans.}$$

7. (a) Find the inverse Laplace transform of : (i) $\frac{3s+1}{s^2(s^2+4)}e^{-3s}$ (ii) $\tan^{-1}\left(\frac{2}{s^2}\right)$.

Solution. (a) (i) $$L^{-1}\left[\frac{3s+1}{s^2(s^2+4)}e^{-3s}\right]$$

By second shifting theoem, $L^{-1}\left\{e^{-as}\bar{f}(s)\right\} = f(t-a)\cdot u(t-a)$

Let $\bar{f}(s) = \frac{3s+1}{s^2(s^2+4)}$ then $f(t) = L^{-1}\left[\frac{3s+1}{s^2(s^2+4)}\right]$

Now $$\frac{3s+1}{s^2(s^2+4)} = \frac{A}{s}+\frac{B}{s^2}+\frac{Cs+D}{(s^2+4)} \qquad ...(1)$$

or $$3s+1 = As^3+4As+Bs^2+4B+Cs^3+Ds^2$$

Comparing Coeff. of s^3, s^2, s and constant term on both sides, we get

$$A+C = 0$$

$$B+D = 0$$

$$4A = 3 \Rightarrow A = \frac{3}{4}$$

$$4B = 1 \Rightarrow B = -\frac{1}{4}$$

$\therefore$ $$A = \frac{3}{4}, B = -\frac{1}{4}, C = \frac{-3}{4}, D = \frac{1}{4}$$

Hence $$L^{-1}\left[\frac{3s+1}{s^2(s^2+4)}\right] = L^{-1}\left[\frac{3}{4s}-\frac{1}{4s^2}-\frac{3s}{4(s^2+4)}+\frac{1}{4(s^2+4)}\right]$$

or $$f(t) = \frac{3}{4}-\frac{t}{4}-\frac{3}{4}\cos 2t+\frac{1}{8}\sin 2t$$

$\therefore \qquad L^{-1}\left\{\frac{2s+1}{s^2(s^2+4)}e^{-3s}\right\} = L^{-1}\left\{e^{-3s}\bar{f}(s)\right\} = f(t-3)\cdot u(t-3)$

$$= \left[\frac{3}{4} - \frac{(t-3)}{4} - \frac{3}{4}\cos 2(t-3) + \frac{1}{8}\sin 2(t-3)\right]\cdot u(t-3)$$ **Ans.**

Solution. (a) (ii) Let $f(t) = L^{-1}\left[\tan^{-1}\left(\frac{2}{s^2}\right)\right]$

Here $\bar{f}(s) = \tan^{-1}\left(\frac{2}{s^2}\right)$

$\therefore \qquad t\,f(t) = L^{-1}\left[-\frac{d}{ds}\left\{\bar{f}(s)\right\}\right] \qquad \left[\because\ f(t) = L^{-1}\left\{\bar{f}(s)\right\}, \text{then } t\,f(t) = L^{-1}\left[\frac{-d}{ds}\left\{\bar{f}(s)\right\}\right]\right]$

or $\qquad t\,f(t) = L^{-1}\left[-\frac{d}{ds}\left\{\tan^{-1}\left(\frac{2}{s^2}\right)\right\}\right]$

$$= L^{-1}\left[-\frac{1}{\left(1+\frac{4}{s^4}\right)}\cdot\left(-\frac{4}{s^3}\right)\right] = L^{-1}\left[\frac{4s}{s^4+4}\right]$$

$$= L^{-1}\left[\frac{4s}{s^4+4s^2+4-4s^2}\right]$$

$$= L^{-1}\left[\frac{4s}{(s^2+2)^2-(2s)^2}\right]$$

$$= L^{-1}\left[\frac{4s}{(s^2+2+2s)(s^2+2-2s)}\right]$$

$$= L^{-1}\left[\frac{(s^2+2s+2)-(s^2-2s+2)}{(s^2+2s+2)(s^2-2s+2)}\right]$$

$$= L^{-1}\left[\frac{1}{(s^2-2s+2)} - \frac{1}{(s^2+2s+2)}\right]$$

$$= L^{-1}\left[\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right]$$

$$= L^{-1}\left[\frac{1}{(s-1)+(1)^2}\right] - L^{-1}\left[\frac{1}{(s+1)^2+(1)^2}\right]$$

$$= e^t\sin t - e^{-t}\sin t = 2\left(\frac{e^t-e^{-t}}{2}\right)\sin t$$

$$= 2\sinh t \sin t \qquad \left[\because \frac{e^t - e^{-t}}{2} = \sinh t\right]$$

$$\therefore \qquad f(t) = \frac{2\sinh t \sin t}{t}. \textbf{ Ans.}$$

7. (b) Solve the partial differential equation $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

Solution. The given partial differential equation is $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

Comparing with $Pp + Qq = R$, we get $P = x^2(y-z), Q = y^2(z-x), R = z^2(x-y)$

$\therefore$ The auxiliary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

or
$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} \qquad ...(1)$$

Taking multipliers $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$ therefore each of the member of equation (1) is

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} = \frac{\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2}}{0}$$

$$\therefore \qquad \frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0$$

On integrating, we get $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = a$ ...(2)

Again taking multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, therefore each of the member of equation (1) is

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

$$\therefore \qquad \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

On integrating we get $\log x + \log y + \log z = \log b$

or $\log(xyz) = \log b$

or $xyz = b$ ...(3)

From (2) and (3), the general solution is $\phi\left[\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, xyz\right] = 0$. **Ans.**

8. (a) Solve the partial differential equation : $z^2 = pq\,xy$

Solution. The given equation is $z^2 = px\,qy$

$$\therefore \qquad f(x,y,z,p,q) = z^2 - px\,qy \qquad \ldots(1)$$

Here $\dfrac{\partial f}{\partial x} = -pqy,\ \dfrac{\partial f}{\partial y} = -pqx,\ \dfrac{\partial f}{\partial z} = 2z,\ \dfrac{\partial f}{\partial p} = -qxy$ and $\dfrac{\partial f}{\partial q} = -pxy$

Charpit's auxiliary equations are

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dF}{0}$$

i.e. $$\frac{dx}{qxy} = \frac{dy}{pxy} = \frac{dz}{pqxy + pqxy} = \frac{dp}{-pqy + 2pz} = \frac{dq}{-pqx + 2qz} = \frac{dF}{0} \qquad \ldots(2)$$

Each fraction in (2) is equal to

$$\frac{p\,dx + x\,dp}{pqxy - pqxy + 2pxz} = \frac{q\,dy + y\,dq}{pqxy - pqxy + 2qzy}$$

i.e. $$\frac{p\,dx + x\,dp}{2pxz} = \frac{q\,dy + y\,dq}{2qzy}$$

i.e., $$\frac{p\,dx + x\,dp}{px} = \frac{q\,dy + y\,dp}{qy} \quad i.e., \quad \frac{d(px)}{px} = \frac{d(qy)}{qy}$$

Integrating $$\log px = \log qy + \log a^2 = \log(a^2 qy)$$

or $$px = a^2 qy \qquad \ldots(3)$$

From (1), $$z^2 = (px)(qy)$$

or $$z^2 = a^2 (qy)^2 \qquad \text{[From (3)]}$$

or $$q^2 = \frac{z^2}{a^2 y^2} \quad \therefore \quad q = \frac{z}{ay}$$

From (3), we get $$p = \frac{a^2 qy}{x} = \frac{a^2 y}{x}\cdot\frac{z}{ay} = \frac{az}{x}$$

Now $$dz = p\,dx + q\,dy$$

Substituting the values of p and q, we have $dz = \dfrac{az}{x}dx + \dfrac{z}{ay}dy$

Integrating, $$\int \frac{dz}{z} = \int \frac{a}{x}dx + \frac{1}{a}\int \frac{1}{y}dy + b$$

i.e. $$\log z = a\log x + \frac{1}{a}\log y + \log b \quad \text{or} \quad \log z = \log(x^a y^{1/a} b)$$

$$z = x^a y^{1/a} b$$ which is the required complete integral. **Ans.**

8. (b) Find the temperature in the thin metal rod of length L, with both the ends insulated (so that there is no passage of heat through the ends) and with initial temperature in the rod $\sin(\pi x / L)$.

Solution. The partial Diff. equation in this case is $\dfrac{\partial^2 u}{\partial x^2} = h^2 \dfrac{\partial u}{\partial t}$ ...(1)

Solution of equation (1) is $u(x,t) = (c_2 \cos px + c_3 \sin px)c_1 e^{-p^2t^2/h^2}$ ...(2)

$\left[\text{Here } h^2 = \dfrac{1}{c^2}\right]$

Putting $x = 0$, $u = 0$ in (2), we get $0 = c_1 c_2 e^{-p^2t^2/h^2}$, $[\because c_1 \neq 0, \therefore c_2 = 0]$

Therefore from (2), we get $u = c_3 \sin(px) c_1 e^{-p^2t/h^2}$ ...(3)

Putting $x = l$, $u = 0$ in equation (3), we get $0 = c_3 \sin(pl) e^{-p^2t/h^2}$

$$\therefore \qquad \sin pl = 0 = \sin n\pi \quad \text{or} \quad p = \frac{n\pi}{l}$$

$\therefore$ Then from equation (3), we have to $u = c_3 \sin\left(\dfrac{n\pi x}{l}\right) e^{n^2\pi^2 t/h^2 l^2}$ ...(4)

Putting $t = 0, u = \sin\dfrac{\pi x}{l}$ in (4) we get $\sin\dfrac{\pi x}{l} = c_3 \sin\dfrac{n\pi x}{l}$. ...(5)

The equation is satisfied if $n = 1$ and $c_3 = 1$.
Putting the values of c_3 and n in (4), we get the required solution as

$$u = \sin\frac{\pi x}{l} e^{-\pi^2 t/h^2 l^2}. \ \textbf{Ans.}$$