B.E. 2nd Sem. Examination 2009-10
Paper : Math-102 - E

Note:- Attempt **five** questions in all, selecting at least **one** question from each part.

PART-A

1. (a) Reduce the given matrix A into normal form and hence find the rank of A, where

$$A = \begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}$$

(b) Determine the values of a and b for which the system

$$x + 2y + 3z = 6$$
$$x + 3y + 5z = 9$$
$$2x + 5y + az = b$$

has
(i) number solution
(ii) unique solution
(iii) infinite no. of solutions.
Find the solution in (ii) and (iii).

2. (a) Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 0 \end{bmatrix}$ Show that the equation is satisfied by A and hence obtain the inverse of the given matrix.

(b) (ii) If λ is an eigen value of an orthogonal matrix, then prove that $\frac{1}{\lambda}$ is also its eigen value.

(iii) Prove that the matrix $\begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$ is orthogonal

PART - B

3. (a) Solve the following differential equation :

$$(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$$

(b) A capacitor C = 0.01 F in series with a resistor R = 20 ohms is charged from a battery

E_0 = 10 V. Assuming that initially the capacitor is completely uncharged, determine the charge Q(t), voltage V (t) and current I (t) in the circuit.

4. (a) Solve the differential equation, $(D^2+4)y = \sin 3x + \cos 2x$

(b) Solve the following differential equation by variation of parameters :

$(D^2-2D+2)y = e^x \tan x$.

5. (a) Solve the following differential equation :

$(1+x)^2 y'' + (1+x)y' + y = 2\sin[\log(1+x)]$

(b) Find the period of a particle of mass m, in simple harmonic motion, attached to the middle point of an elastic string, of natural length 2 a (units) stretched between two points Q and R which are 4 a units apart.

PART - C

6. (a) Solve the following differential equation by using Laplace transformation :

$$y'' + y = t\cos 2t, \quad y(0) = 0, \ y'(0) = 0$$

(b) Using convolution theorem, find the inverse Laplace Transform of the following :

$$\frac{s^2}{(s^2+a^2)(s^2+b)^2}$$

7. (a) Find the Laplace transform of the following functions :

(i) $g(t) = \begin{cases} 0 & , \ 0 < t < 5 \\ t-3 \ , & t < 5 \end{cases}$ (ii) $\dfrac{e^{-at} - e^{-bt}}{t}$

(b) If $L[F(t)] = \bar{f}(s)$. Then prove that the Laplace transform of the function $t^n F(t)$ is

$$(-1)^n \frac{d^n}{ds^n}\bar{f}(s), \ n = 1, 2, 3 \ldots\ldots, \text{ that is, } \quad L[t^n F(t)] = (-1)^n \frac{d^n}{ds^n}\bar{f}(s)$$

8. (a) Solve the following partial differential equation : $z(p-q) = z^2 + (x+y)^2$

(b) Solve the p.d.c. using Charpit's method : $2(z+xp+yq) = yp^2$

SOLUTIONS

PART - A

Solution. 1. (a) $A = \begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}$

operating $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 3 & 6 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -2 & 4 \end{bmatrix}$$

operating $R_1 \to R_1 - 3R_2$, $R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & 9 & -7 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

operating $C_4 \to C_4 + 2C_3,\ C_3 \to C_3 + C_2$

$$\sim \begin{bmatrix} 1 & 0 & 9 & 11 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

operating $C_3 \to C_3 - 9C_1,\ C_4 \to C_4 - 11C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I_2 & : & O \\ .. & & .. \\ O & : & O \end{bmatrix}$$

Which is the required Normal form. $\therefore$ Rank of Matrix A = 2. **Ans.**

Solution. 1. (b)

In matrix notation, the given system of equations can be written $AX = B$

where $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 2 & 5 & a \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 6 \\ 9 \\ b \end{bmatrix}$

$\therefore$ Augmented matrix $[A : B] = \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 1 & 3 & 5 & : & 9 \\ 2 & 5 & a & : & b \end{bmatrix}$

operating $R_2 \to R_2 - R_1,\ R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 0 & 1 & 2 & : & 3 \\ 0 & 1 & (a-6) & : & (b-12) \end{bmatrix}$$

operating $R_1 \to R_1 - 2R_2,\ R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & -1 & : & 0 \\ 0 & 1 & 2 & : & 3 \\ 0 & 0 & (a-8) & : & (b-15) \end{bmatrix}$$

Case I. If $a = 8,\ b \neq 15$

$\rho(A) = 2,\ \rho(A : B) = 3$

$\because \quad \rho(A) \neq \rho(A:B)$

$\therefore$ The system has no solution.

Case II. If $a \neq 8$, b may have any value

$\rho(A) = \rho(A:B) = 3 =$ number of unknowns.

$\therefore$ The system has unique solution.

Case III. If $a = 8$, $b = 15$

$\rho(A) = \rho(A:B) = 2 <$ number of unknowns.

$\therefore$ The system has an infinite number of solutions. **Ans.**

Solution. 2. (a) The characteristic equation of A is

$$|A - \lambda I| = 0 \ \ i.e., \ \ \begin{vmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(2-\lambda)(1-\lambda) - 6] - 3[4(1-\lambda) - 3] + 7[8 - (2-\lambda)] = 0$$

$$\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$$

To verify Cayley – Hamilton theorem, we have to show that

$$A^3 - 4A^2 - 20A - 35I = 0 \qquad \text{....(1)}$$

Now
$$A^2 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$$

$$\therefore \quad A^3 - 4A^2 - 20A - 35I = \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - 4\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$-20\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 35\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

This verifies Cayley - Hamilton theorem.

Now, multiplying both sides of (1) by A^{-1}, we have

$$A^2 - 4A - 20I - 35A^{-1} = O$$

or
$$35A^{-1} = A^2 - 4A - 20I$$

$$= \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 4\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 20\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$$

Hence $$A^{-1} = \frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}.$$ **Ans.**

Solution. 2. (b) (i) Proof : There exists a non-zero vector X such that $AX = \lambda X$

Pre multiplying both sides by A^{-1}, we get

$$A^{-1}(AX) = A^{-1}(\lambda X)$$

$\Rightarrow$ $$(A^{-1}A)X = \lambda(A^{-1}X)$$

$\Rightarrow$ $$X = \lambda(A^{-1}X)$$

$\Rightarrow$ $$\frac{1}{\lambda}X = A^{-1}X \Rightarrow A^{-1}X = \frac{1}{\lambda}X$$

$\Rightarrow$ $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

But $A^{-1} = A'$ $(\because$ A is an orthogonal matrix)

$\therefore$ $\frac{1}{\lambda}$ is on eigen values of A'.

But the matrices A and A' have the same eigen values.

$\therefore$ $\frac{1}{\lambda}$ is also an eigen value of A. **Proved.**

Solution. 2. (b) (ii) Let $$A = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

$\Rightarrow$ $$A' = \begin{bmatrix} -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Now, $$AA' = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} (4+1+4) & (-4+2+2) & (-2-2+4) \\ (-4+2+2) & (4+4+1) & (-4+2+2) \\ (-2-2+4) & (2-4+2) & (4+4+1) \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

$\therefore$ $$A'A = I$$

Hence, A is an orthogonal matrix. **Hence Proved.**

Solution. 3. (a) $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$ ...(1)

Comparing with $Mdx + Ndy = 0$, we have

$$M = y^4 + 2y\ ,\ N = xy^3 + 2y^4 - 4x$$

$\therefore$ $$\frac{\partial M}{\partial y} = 4y^3 + 2 \text{ and } \frac{\partial N}{\partial x} = y^3 - 4$$

Since $\dfrac{\partial M}{\partial x} \neq \dfrac{\partial N}{\partial y}$ thus, eq. (1) is not exact.

Now $$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{y^3 - 4 - 4y^3 - 2}{y^4 + 2y} = \frac{-3}{y} = f(y)$$

$\therefore$ $$I.F. = \int \frac{-3}{y}dy = \frac{1}{y^3}$$

Therefore multiplying (1) by $\dfrac{1}{y^3}$, we get

$$\left(y + \frac{2}{y^2}\right)dx + \left[x + 2y - \frac{4x}{y^3}\right]dy = 0 \quad ...(2)$$

Here $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 1 - \frac{4}{y}$$

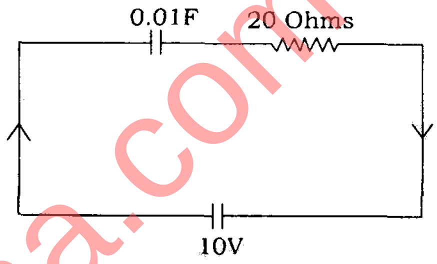
$\therefore$ Equation (2) is exact.

Here required solution is

$$\int_{y\,const}\left(y+\frac{2}{y^2}\right)dx+\int 2y\,dy = C$$

$$\left(y+\frac{2}{y^2}\right)x+y^2 = C$$. **Ans.**

Solution. 3. (b) Here, $iR+\frac{1}{C}\int i\,dt = 10$



Differentiating w.r.t. t

$$\frac{Rdi}{dt}+\frac{i}{C} = 0$$

or $$\frac{di}{dt}+\frac{i}{CR} = 0 \quad ...(1)$$

$\therefore$ $$I.F. = e^{\int\frac{1}{CR}dt} = e^{\frac{t}{CR}}$$

Then solution of (1) is

$$i\cdot e^{t/CR} = A$$

or $$i = Ae^{-t/CR} \quad ...(2)$$

Here $i = 0$, $t = 0$ therefore $A = \frac{V}{R}$

From (2) $$i = \frac{V}{R}e^{-\frac{t}{RC}}$$

Voltage across restor $V_R = Ri = Ve^{-\frac{t}{RC}}$

Voltage across capacitor $V_C = V - Ve^{-t/RC} = 10[1-e^{-t/RC}]$ $[\because V = 10V]$

and Charge $Q = \int i\,dt = \frac{V}{R}e^{-t/RC}$. **Ans.**

Solution. 4. (a) A.E. is $(D^2+4) = 0$

$$D^2 = -4$$

$$D = \pm i = 0 \pm i$$

$\therefore$ $$C.F. = c_1\cos x + c_2\sin x$$

and $$P.I. = \frac{1}{(D^2+4)}(\sin 3x+\cos 2x)$$

$$= \frac{1}{(D^2+4)}(\sin 3x)+\frac{1}{(D^2+4)}(\cos 2x)$$

$$= \frac{\sin 3x}{(-9+4)}+\frac{1}{(-4+4)}(\cos 2x)$$

$$= \frac{\sin 3x}{-5} + \frac{x}{2D}\cos 2x$$

$$= \frac{\sin 3x}{-5} + \frac{x}{2}\int \cos 2x\, dx$$

$$= -\frac{1}{5}\sin 3x + \frac{x}{4}\sin 2x$$

C.S. is $\quad y = C.F. + P.I.$

$$y = c_1 \cos x + c_2 \sin x - \frac{1}{5}\sin 3x + \frac{x}{4}\sin 2x \text{. } \textbf{Ans.}$$

Solution. 4. (b) $\quad (D^2 - 2D + 2)y = e^x \tan x$

A.E. is $\quad D^2 - 2D + 2 = 0$

$$D = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

$\therefore \quad C.F. = e^x[c_1 \cos x + c_2 \sin x]$

Here $\quad y_1 = e^x \cos x,\ y_2 = e^x \sin x$ and $X = e^x \tan x$

$$\therefore \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x \cos x & e^x \sin x \\ (e^x \cos x - e^x \sin x) & (e^x \sin x + e^x \cos x) \end{vmatrix}$$

$$= e^{2x}[\cos x \sin x + \cos^2 x] - e^{2x}[\sin x \cos x - \sin^2 x]$$

$$= e^{2x}[\cos x \sin x + \cos^2 x - \sin x \cos x + \sin^2 x]$$

$$= e^{2x}$$

Now $\quad P.I. = -y_1 \int \frac{y_2 X}{W}\, dx + y_2 \int \frac{y_1 X}{W}\, dx$

$$= -e^x \cos x \int \frac{e^x \sin x \cdot e^x \tan x}{e^{2x}}\, dx + e^x \sin x \int \frac{e^x \cos x \cdot e^x \tan x}{e^{2x}}$$

$$= -e^x \cos x \int \sin x \tan x\, dx + e^x \sin x \int \cos x \tan x\, dx$$

$$= -e^x \cos x[\log(\sec x + \tan x) - \sin x] - e^x \sin x \cos x$$

$$= -e^x \cos x \log(\sec x + \tan x) + e^x \sin x \cos x - e^x \sin x \cos x$$

$$= -e^x \cos x \log(\sec x + \tan x)$$

Hence C.S. is $\quad y = C.F. + P.I.$

$$y = e^x(c_1 \cos x + c_2 \sin x) - e^x \cos x \log(\sec x + \tan x) \text{. } \textbf{Ans.}$$

Solution. 5. (a) Given equation is a Legendre's linear equation.

Put $(1+x) = e^z$ *i.e.*, $z = \log(1+x)$

so that $(1+x)y' = Dy$, $(x+1)^2 y'' = D(D-1)y$, where $D = \dfrac{d}{dz}$

Substituting these values in the given equation, it reduces to

$$[D(D-1)+D+1]y = 2\sin z$$

$$(D^2 - D + D + 1)y = 2\sin z$$

$$(D^2+1)y = 2\sin z$$

Which is a linear equation with constant co-efficients.

Its A.E. is $D^2 + 1 = 0$

$\therefore$ $D = \pm\, i$

$\therefore$ $C.F. = c_1 \cos z + c_2 \sin z$

$$P.I. = \frac{1}{(D^2+1)}(2\sin z)$$

$$= \frac{2\sin z}{(-1+1)} \quad \text{(Case fail)}$$

$\therefore$ $$P.I. = z \cdot \frac{1}{2D}(2\sin z)$$

$$= z \cdot \frac{1}{D}(\sin z) = -z\cos z$$

Hence the C.S. is $y = C.F. + P.I.$

$$y = c_1 \cos z + c_2 \sin z - z\cos z$$

or $$y = c_1 \cos[\log(1+x)] + c_2 \sin[\log(1+x)] - \log(1+x)\cos[\log(1+x)]$$. **Ans.**

Solution. 5. (b) Out of syllabus.

Solution. 6. (a) Here, $y'' + y = t\cos 2t$

Taking Laplace transform of both sides, we get

$$L[y''] + L[y] = L[t\cos 2t]$$

$$[s^2\bar{y} - sy(0) - y'(0)] + \bar{y} = L[t\cos 2t]$$

$$[s^2\bar{y} - 0 - 0] + \bar{y} = -\frac{d}{ds}\left[\frac{s}{s^2+4}\right]$$

$$(s^2+1)\bar{y} = -\left[\frac{s^2+4-2s^2}{(s^2+4)^2}\right] = -\left[\frac{4-s^2}{(s^2+4)^2}\right]$$

or $$\bar{y} = \frac{s^2-4}{(s^2+4)^2(s^2+1)}$$

Taking inverse Laplace transform of both sides

$$y = L^{-1}\left[\frac{s^2-4}{(s^2+4)^2(s^2+1)}\right] \quad \ldots(1)$$

Let
$$\frac{s^2-4}{(s^2+4)^2(s^2+1)} = \frac{z-4}{(z+4)^2(z+1)}, \text{ where } s^2 = z \quad \ldots(2)$$

Now,
$$\frac{z-4}{(z+1)(z+4)^2} = \frac{A}{(z+1)} + \frac{B}{(z+4)} + \frac{C}{(z+4)^2} \quad \ldots(3)$$

$$\therefore \quad z-4 = A(z+4)^2 + B(z+1)(z+4) + C(z+1) \quad \ldots(4)$$

Putting $z = -1$ in (4), $-5 = 9A$, $\quad \therefore A = -\frac{5}{9}$

Putting $z = -4$ in (4), $-8 = -3C \Rightarrow C = \frac{8}{3}$

Comparing the constant terms in (4), we get

$$-4 = 16A + 4B + C$$

$$-4 = \frac{-80}{9} + 4B + \frac{8}{3}$$

$$\frac{-36-24+80}{9} = 4B \Rightarrow B = \frac{5}{9}$$

$$\therefore \quad \frac{s^2-4}{(s^2+4)^2(s^2+1)} = \frac{-5}{9}\frac{1}{(s^2+1)} + \frac{5}{9}\frac{1}{(s^2+4)} + \frac{8}{3}\cdot\frac{1}{(s^2+4)^2}$$

$$\therefore \quad L^{-1}\left[\frac{s^2-4}{(S^2+4)^2(s^2+1)}\right] = \frac{-5}{9}L^{-1}\left[\frac{1}{s^2+1}\right] + \frac{5}{9}L^{-1}\left[\frac{1}{s^2+4}\right] + \frac{8}{3}L^{-1}\left[\frac{1}{(s^2+4)^2}\right]$$

$$= \frac{-5}{9}\sin t + \frac{5}{9}\cdot\frac{1}{2}\sin 2t + \frac{8}{3}\cdot\frac{1}{2\times 8}(\sin 2t - 2t\cos 2t)$$

$$\left[\therefore \quad L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \frac{1}{2a^3}(\sin at - at\cos at)\right]$$

$$= \frac{-5}{9}\sin t + \frac{5}{18}\sin 2t + \frac{1}{6}\sin 2t - \frac{1}{3}t\cos 2t$$

$\therefore$ From (1)
$$y = -\frac{5}{9}\sin t + \frac{4}{9}\sin 2t - \frac{1}{3}t\cos 2t$$

Hence
$$y = \frac{1}{9}[4\sin 2t - 5\sin t - 3t\cos 2t].$$ **Ans.**

Solution. 6. (b)
$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\left(\frac{s}{s^2+a^2}\right)\cdot\left(\frac{s}{s^2+b^2}\right)\right] \quad \ldots(1)$$

Let
$$L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at = f(t) \text{ and } L^{-1}\left[\frac{s}{s^2+b^2}\right] = \cos bt = g(t)$$

$\therefore$ From (1) by the convolution theorem, we have

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\left(\frac{s}{s^2+a^2}\right)\cdot\left(\frac{s}{s^2+b^2}\right)\right]$$

$$= \int_0^t f(u)\cdot g(t-u)\,du$$

$$= \int_0^t \cos au\cdot\cos b(t-u)\,du$$

$$= \frac{1}{2}\int_0^t [2\cos au\cdot\cos(bt-bu)]\,du$$

$$= \frac{1}{2}\int_0^t [\cos(au+bt-bu)+(\cos(au-bt+bu)]\,du$$

$$[\because\ 2\cos A\cos B = \cos(A+B)+\cos(A-B)]$$

$$= \frac{1}{2}\int_0^t \left[\cos\{(a-b)u+bt\}+\cos\{(a+b)u-bt\}\right]du$$

$$= \frac{1}{2}\left[\frac{\sin\{(a-b)u+bt\}}{(a-b)}+\frac{\sin\{(a+b)u-bt\}}{(a+b)}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin(at-bt+bt)}{a-b}+\frac{\sin(at+bt-bt)}{a+b}-\frac{\sin bt}{a-b}-\frac{\sin(-bt)}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b}+\frac{\sin at}{a+b}-\frac{\sin bt}{a-b}+\frac{\sin bt}{a+b}\right] \quad [\because\ \sin(-\theta) = -\sin\theta]$$

$$= \frac{1}{2}\left[\frac{(a+b+a-b)\sin at}{(a+b)(a-b)}+\frac{(a-b-a-b)\sin bt}{(a+b)(a-b)}\right]$$

$$= \left[\frac{a\sin at}{a^2-b^2}-\frac{b\sin bt}{a^2-b^2}\right]$$

$$= \frac{1}{(a^2-b^2)}[a\sin at-b\sin bt].$$ **Ans.**

Solution. 7. (a) (i)

$$g(t) = \begin{cases} 0, & 0<t<5 \\ t-3, & t>5 \end{cases}$$

Here

$$L[f(t)] = \int_0^\infty e^{-st}f(t)\,dt$$

$$= \int_0^5 e^{-st}\cdot 0\,dt+\int_5^\infty e^{-st}(t-3)\,dt$$

$$= 0+\left[(t-3)\frac{e^{-st}}{-s}-\int e^{-st}\,dt\right]_5^\infty$$

$$= \left[(t-3)\frac{e^{-st}}{-s} + \frac{1}{s}e^{-st}\right]_5^{\infty}$$

$$= \left[(5-3)\frac{e^{-5s}}{-s} + \frac{1}{s}e^{-5s}\right]$$

$$= \frac{2e^{-5s}}{s} - \frac{1}{s}e^{-5s} = \frac{e^{-5s}}{s} \quad \textbf{Ans.}$$

Solution. 7. (a) (ii) $L\left[\frac{e^{-at} - e^{-bt}}{t}\right]$

$$= L\left[\frac{e^{-at}}{t}\right] - L\left[\frac{e^{-bt}}{t}\right]$$

$$= \int_s^{\infty} \frac{1}{s+a} ds - \int_s^{\infty} \frac{1}{s+b} ds$$

$$= [\log(s+a) - \log(s+b)]_s^{\infty}$$

$$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^{\infty} = \left[\log\left[\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right]\right]_s^{\infty}$$

$$= 0 - \log\left[\frac{s+a}{s+b}\right] = \log\left[\frac{s+b}{s+a}\right]. \quad \textbf{Ans.}$$

Solution. 8. (a) The given equation may be written as

$$pz - qz = z^2 + (x+y)^2$$

Comparing it with $Pp + Qq = R$, we have $P = z, Q = -z, R = z^2 + (x+y)^2$

Subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

i.e., $$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x+y)^2} \quad \ldots(1)$$

Taking the first two fractions of (1),

$$dx = -dy$$

Integrating $$x = -y + a \quad \ldots(2)$$

or $$x + y = a$$

Taking the 2nd and 3rd fractions of (1),

$$\frac{dy}{-z} = \frac{dz}{z^2 + (x+y)^2}$$

or $$dy = -z\frac{dz}{z^2+a^2}$$ [Using (2)]

Integrating, $$\int dy = -\frac{1}{2}\int\frac{2z}{z^2+a^2}dz + c$$

i.e., $$y = -\frac{1}{2}\log(z^2+a^2)+c$$

or $$2y = -\log(z^2+a^2)+\log b \quad \text{(say)}$$

or $$2y = \log\frac{b}{z^2+a^2}$$

$\therefore$ $$b = e^{2y}(z^2+a^2) = e^{2y}[z^2+(x+y)^2]$$ $[\because a = x+y]$

Hence the solution is $e^{2y}[z^2+(x+y)^2] = f(x+y)$. **Ans.**

Solution. 8. (b) Here, $f(x,y,z,p,q) = 2z+2xp+qy-yp^2 = 0$ ...(1)

$\therefore$ $$\frac{\partial f}{\partial x} = 2p,\ \frac{\partial f}{\partial y} = 2q-p^2,\ \frac{\partial f}{\partial z} = 2,\ \frac{\partial f}{\partial p} = 2x-2py,\ \frac{\partial f}{\partial q} = 2y$$

Charpits auxilary equations are

$$\frac{dx}{\frac{-\partial f}{\partial p}} = \frac{dy}{\frac{-\partial f}{\partial q}} = \frac{dz}{-p\frac{-\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dF}{0}$$

$$\frac{dx}{-2x+2py} = \frac{dy}{-2y} = \frac{dz}{-p(2x-2py)-2qy} = \frac{dp}{4p} = \frac{dq}{4q-p^2} = \frac{dF}{0}$$

Taking 2nd and 4th members, we have $\frac{dy}{-2y} = \frac{dp}{4p}$ or $\frac{dp}{p}+2\frac{dy}{y} = 0$

on integrating $\log p + 2\log y = \log a$ or $a = py^2$ or $p = \frac{a}{y^2}$

These values putting in (1), we get $q = \frac{a^2}{2y^4}-\frac{ax}{y^3}-\frac{z}{y}$

Now $dz = pdx+qdy$ or $dz = \frac{a}{y^2}dx+\left[\frac{a^2}{2y^4}-\frac{ax}{y^3}-\frac{z}{y}\right]dy$

$$ydz+zdy = a\left[\frac{ydx-xdy}{y^2}\right]+\frac{a^2}{2y^3}dy$$

$$d(yz) = ad\left[\frac{x}{y}\right]+\frac{a^2}{2y^3}dy$$

on integrating $$yz = \frac{ax}{y}-\frac{a^2}{4y^2}+b \quad \text{or} \quad z = \frac{ax}{y^2}-\frac{a^2}{4y^3}+\frac{b}{y}.$$ **Ans.**