$$\boxed{\cancel{a+b = a+c \Rightarrow b=c}}$$

But

$$\boxed{\begin{array}{l} ab = ac \\ \& \ a+b = a+c \Rightarrow b = c \end{array}}$$

Date 04/09/2017

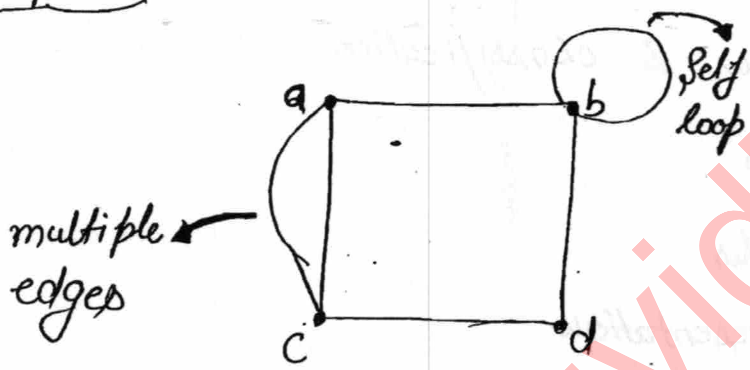
Graph Theory :-

1. Graph definition & classification
★ 2. Vertex degrees
★ 3. Special graphs
4. Graph representation
5. Graph operations $G_1 \cup G_2$, $G_1 \cap G_2$, $G_1 - G_2$, $\bar{G}$, $G_1 \oplus G_2$
6. Isomorphism
7. Connectivity of graphs :-
   i) connected & disconnected graph
   ii) cut vertex, cut edge, cut set (weak points of graph)
   iii) vertex connectivity, edge connectivity
   iv) Strongly connected : SC
   unilaterly connected : UC
   weakly connected : WC
   disconnected : DC
   } digraph
8. Application :
   1. Euler graphs, Hamiltonian graph
   2. Planner graph
   3. Trees. 4. Enumeration of graphs 5. Graph No's

Graph No's :

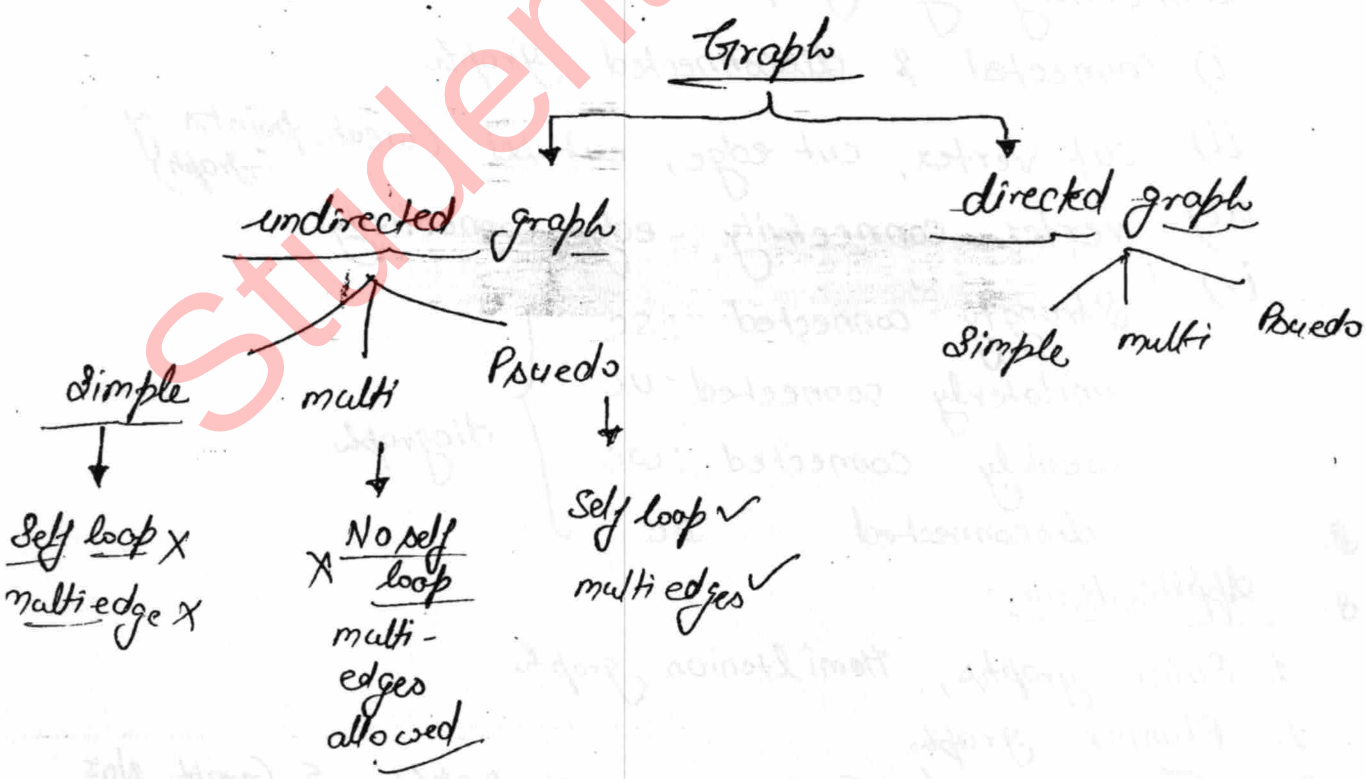
1. chromatic number
2. Independent number
3. Dominant number
4. Matching number
5. Covering number

:- "Graph" :-



i) undirected graph (By default)

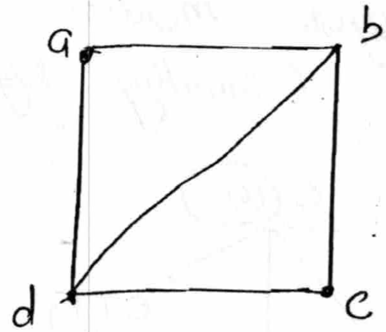
ii) Directed graph (Di-graph)



:- $G(V, E)$

$V = \{a, b, c, d\}$

$e = \{\{a,b\}, \{b,c\}, \{c,d\}, \{d,b\}, \{b,d\}\}$



:- $|V| \geq 1$ i.e. vertex set can't be empty in graph

:- $|e| \geq 0$

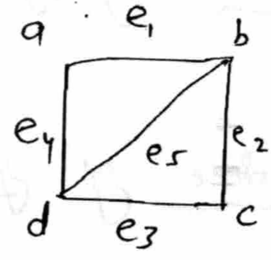
:- Null graph :- No edge :: (vertex are there).

:- In Edge labeled graph, we need a mapping function also, $G(V, E, F)$

$f(e_1) = \{a, b\}$

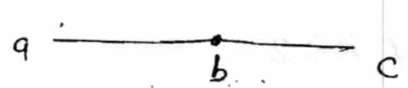
$f(e_2) = \{b, c\}$

$F_E \rightarrow V \times V$



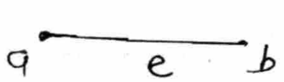
$\Rightarrow$ adjacent vertex :- To vertices are adjacent if they are directly connected.

:-



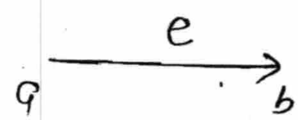
(a, c) are not adjacent

$\Rightarrow$



e is incident upon a & b.

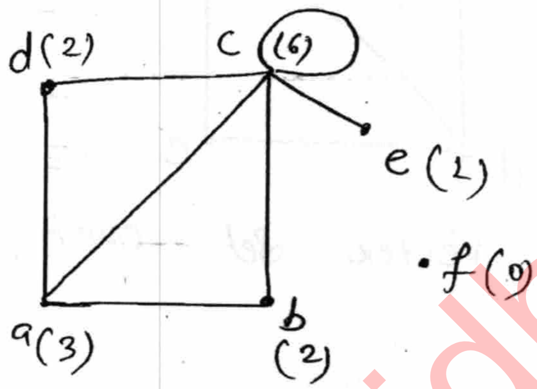
In directed edge :-



:- Degree of A Vertex :-

Deg (v) = # of edges incident upon 'v'
(counting self loop twice)

eg :-



degree (v) = 0 :- 'isolated vertex' or Lone vertex.

degree (v) = 1 :- 'Pendant vertex' or leaf (tree)

order of graph (G) ⇒ # of virtices in graph
= n = 6

size of graph (e) :- # edges, (self loop (1 edge))
= e = 7

# of components :- G(k) = 2

★ graph is 'connected iff k = 1'

★ graph is disconnected iff k ≥ 2

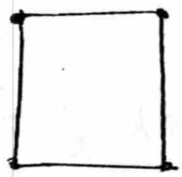
★ δ = minimum degree of graph

$\delta(G) = 0$

★ Δ = maximum degree of graph

$\Delta(G) = \underline{6}$

★ If $\delta = \Delta$ in a graph then graph is called as '<u>Regular graph</u>'.



→ 2 Regular graph

$\delta = \Delta = k$ k-Regular graph

<u>Theorem</u> :–

1. Hand Shaking theorm → $\boxed{\sum_{\forall v \in G} deg.(v) = 2e}$
2. Havell- Hakimi theorem

:– Degree sequence given :–

$(0, 1, 2, 2, 3, 6)$

$n = 6$

e = By Hand-shaking theorm

$e = \dfrac{\sum d}{2} = \dfrac{14}{2} = \underline{7}$

:– mipping degree

$(0, 1, 2, 2, x, 6)$ $e = 7$ given

$0 + 1 + 2 + 2 + x + 6 = 2 \times e$

$\boxed{x = 3}$

$(0, 1, 2, 2, x, y)$

$0 + 1 + 2 + 2 + x + y = 2e \Rightarrow \boxed{x + y = 9}$

eg:- If in a Tree G Every vertex has degree 1, 2, 3, .. k and no of vertex degree $2 = n_2$

# of vertex of degree $3 = n_3$

⋮

degree $k = n_k$

# of leaf node = ?

i.e. # of vertex of deg(1) = let x

Soln, -

$$1 \times x + 2 \times n_2 + 3 \times n_3 + \cdots + k n_k = 2e$$

for tree with n vertices, edges $= n-1$

$$1 \times x + 2n_2 + 3n_3 + \cdots + kn_k = 2(x + n_2 + n_3 + \cdots + n_k - 1)$$

$$\boxed{x = n_3 + 2n_4 + 3n_5 + \cdots + (k-2)n_k + 2}$$

★ 1.) No graph can have odd no of vertices of odd degree.

★ 2.) In Every graph # of vertices of odd degree is always even.

★ 3.) max No. edges in a simple graph with n vertices $= \dfrac{n(n-1)}{2}$

Simple graph with max no. of edges is also called as complete graph.

# of edges in complete graph $(K_n) = \dfrac{n(n-1)}{2}$

★ K- Regular graph ~~has~~ of order n has # of edges $\boxed{e = \frac{nk}{2}}$

★ A n-vertices complete graph is always also called $(n-1)$ regular graph.

★ No graph of odd order can be k-regular, when k is odd.

for Directed graph :-

$$\boxed{\sum_{\forall v \in G} in\ deg(v) = \sum_{\forall v \in G} out\ deg(v) = e}$$

$$\sum deg^-(v) = \sum deg^+(v) = e$$

eg:- $e = 5$ (directed graph)

(1, 1), (2, 2), (2, x), (y, 1)

$1 + 2 + 2 + y = e = 5$

$5 + y = 5$

$y = 0$

$1 + 2 + x + 1 = 5$

$x = 1$

Havell - Hakimi theorm :-

→ graphical Sequence : Degree sequence of simple graph

A degree seq $(d_1, d_2, d_3 \ldots, d_n)$
(In decreasing order)

If not given then, put it into decreasing order.

eg:- $(\cancel{5}, 5, 4\,4, 3\,3, 2, 1, 1)$

$(\cancel{4}, 3, 3, 2, 2, 2, 1, 1)$

$(2, 2, 1, 1, 2, 1, 1, 1)$

$(\cancel{2}, 2, 2, 1, 1, 1, 1)$

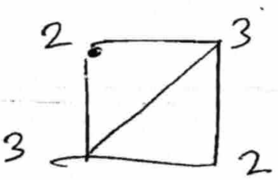
$(1, 1, 0, 1, 1, 1, 1)$ ← even 1's

★ In a Graphical sequence atleast one degree must be repeated.

⇒ No graphical sequence is possible with no repeation of degree.

★
(★
3.) δ, Δ then $\boxed{\delta \le \frac{2e}{n} \le \Delta}$

$\frac{2e}{n}$ ⇒ Average degree per vertex.



$2 \le \frac{10}{4} \le 3$

$2 \le 2.5 \le 3$

eg:- graph of size 16 and order 10

size = edge = e = 16
order = vertices = n = 10

$\delta \le ?$

$$\delta \le \frac{2e}{n}$$

$$\delta \le \left\lfloor \frac{2 \times 16}{10} \right\rfloor$$

$$\delta = 3$$

$$\Delta \ge \lceil 3.2 \rceil$$

$$\Delta = 4$$

eg:- $n = 10, \quad \delta = 3$

$$\delta \le \frac{2e}{n}$$

$$3 \le \frac{2e}{10}$$

$$2e \ge 30$$

$$e \ge 15$$

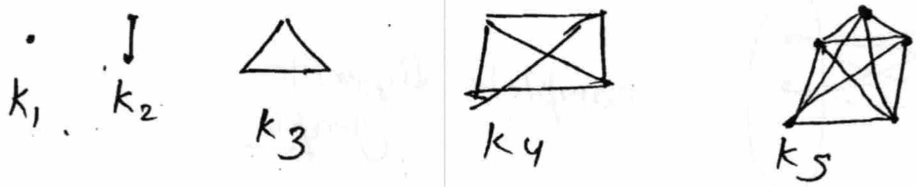
Special graphs :-

1. Null graph
2. Regular graph
3. Complete graph
4. Cycle graph
5. wheel graph
6. n - cubes
7. Bipartite graph
8. complete Bipartite graph

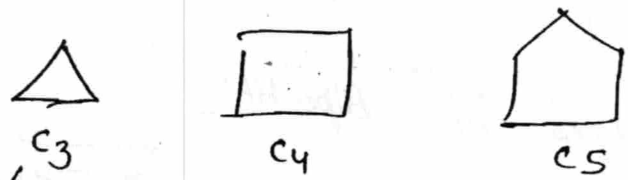
Special graphs	notation	n	$e = f(n)$	Def^n	chromatic No. (colouring) K_G	Euler	Hamiltonian	Diameter
1. Null graph	$\phi(n)$	n	0	$\lvert E \rvert = 0$	1	×	×	∞
2. Regular graph	K-regular	—	$\frac{nk}{2}$	—	—	(connected + K = even)	— ×	—
3. complete graph	K_n	n	$\frac{n(n-1)}{2}$	S.G. with max no. of edges	n	$n \geq 3$ & $n \to$ odd	$n \geq 3$ ✓	1
4. cycle graph	C_n	n	n	Polygon	$K(C_n) = 2$, n even; $K(C_n) = 3$, n odd; $(2 + n \bmod 2)$	✓	✓	$\left\lfloor \frac{n}{2} \right\rfloor$
5. wheel graph	W_n	n	$2(n-1)$		$K(W_n) = 3$, n odd; $K(W_n) = 4$, n even; $= 3 + (n-1) \bmod 2$	×	✓	$W_4 =$ [illegible]; $W_{n \geq 5} =$ [illegible]
6. n-cubes	Q_n, $n \geq 1$	2^n	$n \cdot 2^{n-1}$	→ Boolean algebra	2	n = even	Q_1 ×, Q_2 ✓, Q_3 ✓, others —	n
7. Bipartite	—			→ G is B.P. iff it has no odd cycle	2 (non null)	—	—	—
8. complete Bipartite graph	$K_{m,n}$	$m+n$	$m \cdot n$	Bipartite + complete	2	m, n both even	—	2

$Q_n \to$ always Bipartite & n-regular

Complete graphs :-



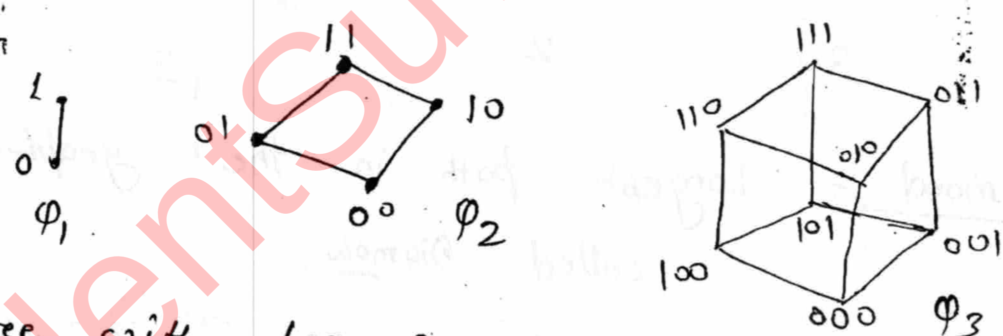
cyclic graph :-



C_3 → only complete

wheel graph :

n-cubes :-



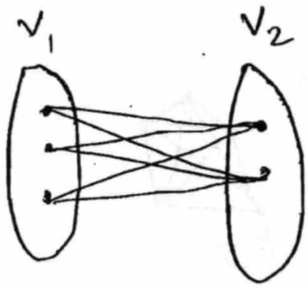
→ Every tree with two or more vertices is Bipertite.

→ Every Acyclic graph is Bipertite.

→ In Biperfite graph, no need of more than 2 colors.

→ Non null - Bicromatic graph $\Leftrightarrow$ Bipertite graph

→ Q_n (n-cube) always Bipertite so Bichromatic i.e. chromatic no = 2



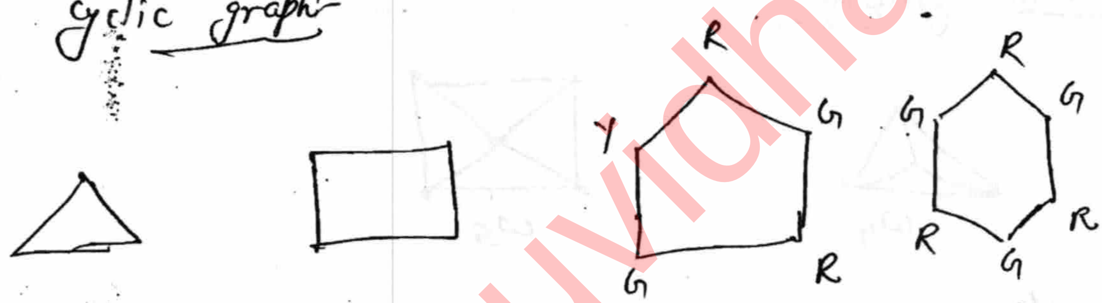
complete Bipertite graph.

K(3, 2)
K(m, n)

For max edges in Bipertite

$e_{max} = m.n$ when $m = n$

Colouring cyclic graph:-



3 2 3 2

Diamond :- Longest path in the graph is called Diamoh.

path :- shortest distance between two nodes.

Diamond of cyclic graph:-

C_3	C_4	C_5	C_6
1	2	2	3

So $\left\lfloor \frac{n}{2} \right\rfloor$

Ratio of chromatic number and diamond :-
K (m,n) (Complete Bipertite)

$$\Rightarrow \frac{\text{chromatic no}}{\text{diamond}} = \frac{2}{2}$$

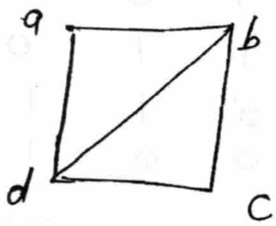
Graph Representation :-

1. Adjacency matrix (edge unlabeled)
2. Incidence matrix (used only in edge labelled graphs)
3. Adjacency list (in multigraph, It can't be used)
only for simple graph

i) Adjacency matrix :-

$$M_G = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 0 & 1 \\ b & 1 & 0 & 1 & 1 \\ c & 0 & 1 & 0 & 1 \\ d & 1 & 1 & 1 & 0 \end{array}$$

$|V| \times |V|$ $n \times n$



$V = \{a, b, c, d\}$

Simple graph :- all diagonal element '0'
& matrix = Boolean matrix (0, 1)

multi-graph :- all diagonal element '0'

Q eg :- $A_G = [a_{ij}]$

$$\sum_{\forall j} \sum_{\forall i} a_{ij} = ?$$

$$\begin{array}{c|cccc|l} & a & b & c & d & \\ \hline a & 0 & 1 & 0 & 1 & 2 \rightarrow deg.(a) \\ b & 1 & 0 & 1 & 1 & 3 \quad deg(b) \\ c & 0 & 1 & 0 & 1 & 2 \quad deg(c) \\ d & 1 & 1 & 1 & 0 & 3 \quad deg(d) \\ \hline & 2 & 3 & 2 & 3 & 10 = 2e \end{array}$$

In directed graph

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 0 & 1 \\ b & 0 & 0 & 1 & 1 \\ c & 0 & 1 & 0 & 1 \\ d & 1 & 1 & 0 & 0 \end{array}$$

2 → out degree

in degree

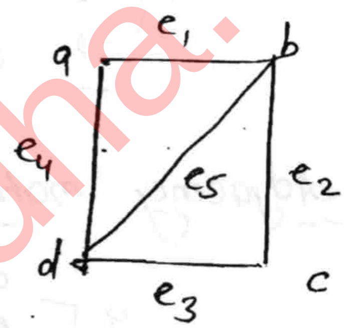
:- In directed graph, Row summation → out degree

Coloum summation → in degree.

:- Incidence matrix :-

$$\begin{array}{c|ccccc} & e_1 & e_2 & e_3 & e_4 & e_5 \\ \hline a & 1 & 0 & 0 & 1 & 0 \\ b & 1 & 1 & 0 & 0 & 1 \\ c & 0 & 1 & 1 & 0 & 0 \\ d & 0 & 0 & 1 & 1 & 1 \end{array}$$

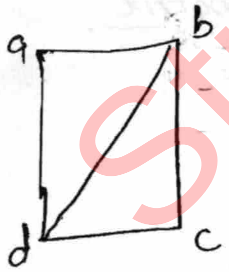
→ deg(a)



size = $|V| \times |E|$ → (not square matrix always)

Row summation gives degree of vertex.

:- Adjacency list :-

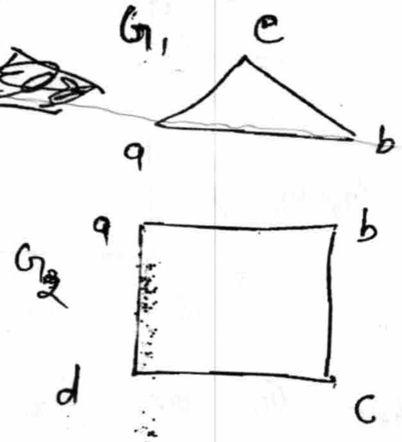


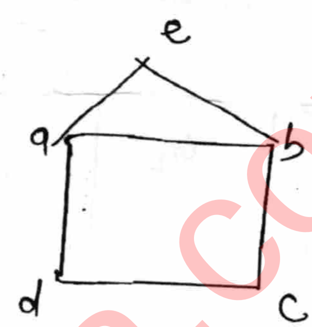
V	adjacency vertex
a	b, d
b	a, c, d
c	b, d
d	a, c, b

Graph operations :-

1) $G_1 \cup G_2$:-

take union of vertices, and edges





2) $G_1 \cap G_2$

$$V(G_1 \cap G_2) = V(G_1) \cap V(G_2)$$

$$E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$$

:- No vertex in common → vertex disjoint

:- No edge in common → edge disjoint

vertex disjoint ⇒ edge disjoint i-e

:- every vertex disjoint also be edge disjoint.

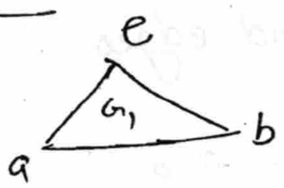
:- ★ Intersection of graph may not exist when the graph is vertex disjoint.

$$\boxed{\bar{G} = K_n - G}$$

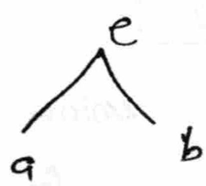
universal → complete graph

:- In subtraction, subtract common edges.

<u>$G_1 - G_2$:-</u>



$G_1 - G_2 \Rightarrow$



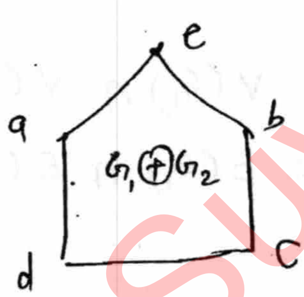
$G_2 - G_1 \Rightarrow$

<u>$G_1 \oplus G_2 \Rightarrow$</u> . edges to either G_1 or G_2 but not Both.

(Symmetric difference)

or

'<u>Ring-sum</u>'



eg:- given :- $|V(G_1)| = 10$ $|E(G_1)| = 5$ $|E(G_1 \cap G_2)| = 2$

$|V(G_2)| = 5$ $|E(G_2)| = 6$

$|V(G_1 \oplus G_2)| =$

$|E(G_1 \oplus G_2)| =$

$$V(G_1 \oplus G_2) = V(G_1 \cup G_2) = V(G_1) + V(G_2) - V(G_1 \cap G_2)$$
$$= 10 + 5 - 3$$
$$= 12$$

$$E(G_1 \oplus G_2) = E(G_1) + E(G_2) - 2E(G_1 \cap G_2)$$
$$= 5 + 6 - 2 \times 2$$
$$= 11 - 4 = 7$$

$|V((G_1) - G_2))| = 10$

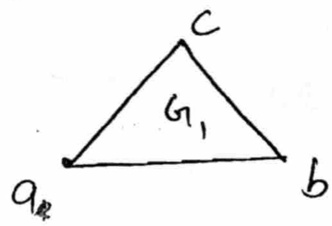
$|E(G_1 - G_2)| = 5 - 2 = 3$

$G_1 \cap G_2$

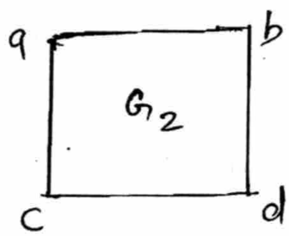
$|V(G_2 - G_1)| = 5$

$|E(G_2 - G_1)| = 6 - 2 = 4$

:- $\overline{G}$

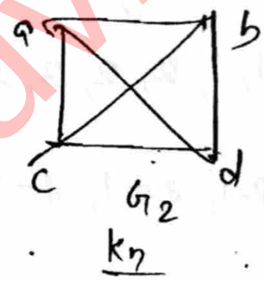


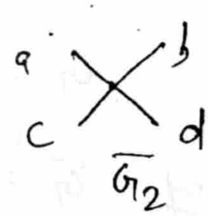
$\overline{G_1} = K_n - G_1$

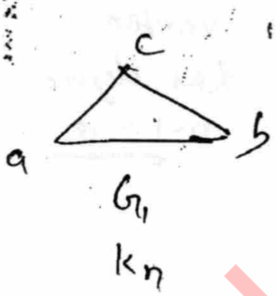


$\overline{G_2} = K_n - G_2$

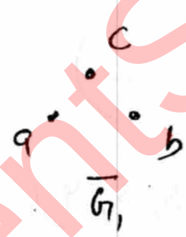
# of virtices







$\Rightarrow$



$\Rightarrow \overline{K_n} = \phi_n$

$\overline{\phi_n} = K_n$

$G \cup \overline{G} = K_n$

$G \cap \overline{G} = \phi_n$

eg:- Graph has Order 10

$G \cup \overline{G} = K_{10}$

$G \cap \overline{G} = \phi_{10}$

$V(\bar{G}) = V(G)$

~~$E(\bar{G}) = E(G)$~~

$E(\bar{G}) = E(K_n) - E(G)$

$|V(\bar{G})| = |V(G)|$

$$\boxed{|E(\bar{G})| = \frac{n(n-1)}{2} - E(G)}$$

eg:- Graph of order 8

If $E(\bar{G}) = E(G)$

$x = \frac{8 \times 7}{2} = \frac{56}{2} = 28$

$E(G) = 28 - x$

$x = 14$

degree sequence of graph

$G = (1, 1, 2, 2, 3, 3, 3, 3, 4) \Rightarrow k_9$ each vertex has degree $n-1 = 8$

degree sequence of $\bar{G} = (8-1, 8-1, 8-2, \ldots)$

$\bar{G} = (7, 7, 6, 6, 5, 5, 5, 5, 4)$

$\bar{G} = (4, 5\ 5, 5, 5, 6, 6, 7, 7)$

(or)

degree sequence of $G = (d_1\ d_2, d_3 \ldots d_n)$

then deg sequence of $\bar{G} = (n-1-d_n, n-1-d_{n-1}, \ldots n-1-d_1)$

Self Complimentry graph :-

∴ Atleast one of g or $\bar{g}$ must be connected

G	$\bar{G}$
C	C
C	DC
DC	C
DC	DC ✗

→ G is C ⇒ $\bar{G}$ may or may not connected

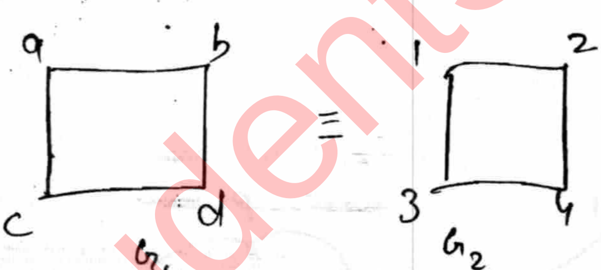
→ $\bar{G}$ is C ⇒ $\bar{\bar{G}}$ "

→ G is DC ⇒ $\bar{G}$ has to be connected

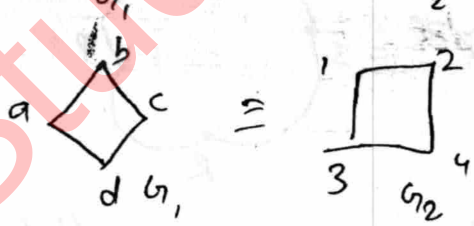
→ $\bar{G}$ is DC ⇒ G has to be connected.

A Graph G is self complimentry if & only if G_1 is isomorphic to $\bar{G}$ i.e

$$G \cong \bar{G}$$

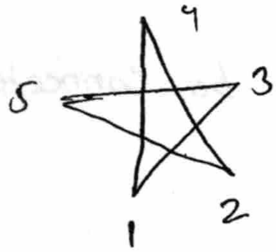


G_1 & G_2 are isomorphic.

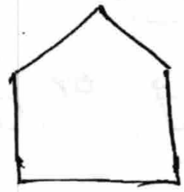


$\bar{G}$ ∴ Not self compliment

self complimented

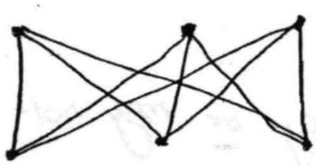


$\equiv$

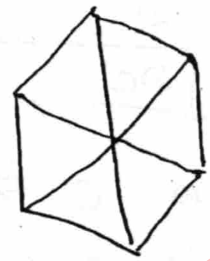


self complimented graph

Isomorphism



$\equiv$



Not self complimented

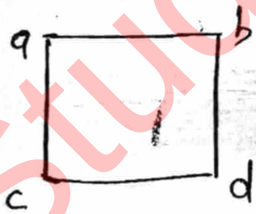
*** 1.) G is self complimented $\Rightarrow$ # of vertices in the form of $4x$ or $4x+1$ (one way)

i.e $n \bmod 4 = 0$ or 1

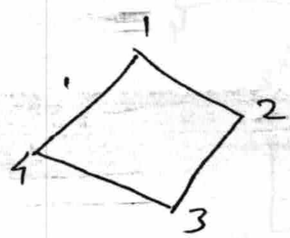
*** 2.) G is self complimented $\Rightarrow$ $e = \dfrac{n(n-1)}{4}$

+ Isomorphism :-

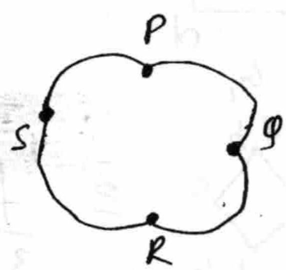
$G_1 \equiv G_2$



$\equiv$



$\equiv$



$G_1 \equiv G_2$ Iff $\exists$ Bijection $f_{V(G_1)} \rightarrow V(G_2)$ which 'preserves bijection' i.e.

a is adjacent to b Iff $f(a)$ adjacent to $f(b)$ in G_2
in G_1

:- $G_1 \equiv G_2$ iff $M_{G_1} \equiv M_{G_2}$ for some ordering.
→ Adjacency matrix

If $G_1 \equiv G_2$, The following are Invariants (similiarities)

:- 1. $n(G_1) = n(G_2)$

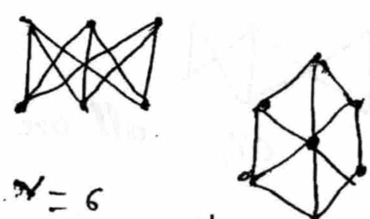
2. $e(G_1) = e(G_2)$

3. degree seq. (G_1) = degree seq. (G_2)

4. # cycles of any length same G_1 & G_2

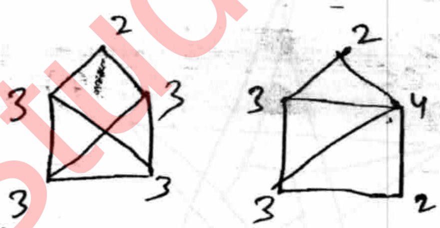
5. Corresponding vertices in Both graph
in v_1 G_1 v_2 G_2
quality of neighbours must be same
↓
in terms of degree

Not isomorphic

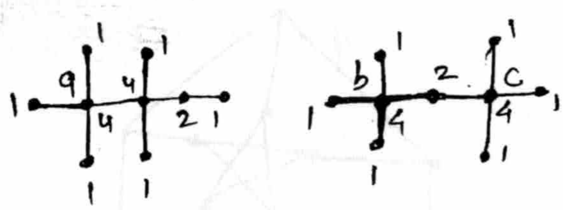


V = 6

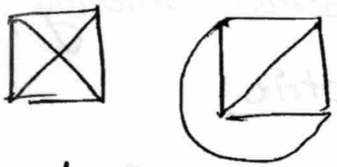
Not isomorphic
V = 7



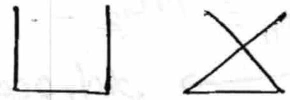
deg sequence is not sam.
$(2,3,3,3,3)$ $G_2(2,2,3,3,4)$
so, Not isomorphic



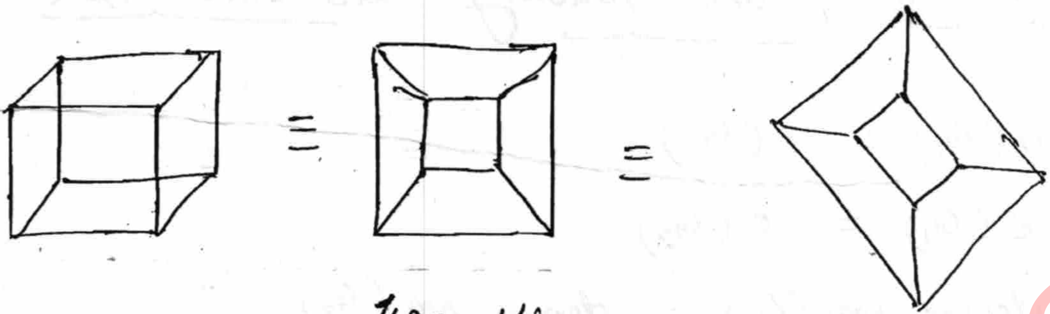
But Not isomorphic
because quality of neighbours is not same
b, c both not having any 4 deg. vertex but a in G_1 has



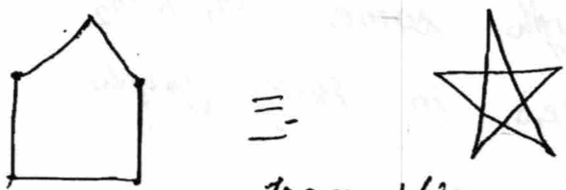
Isomorphic



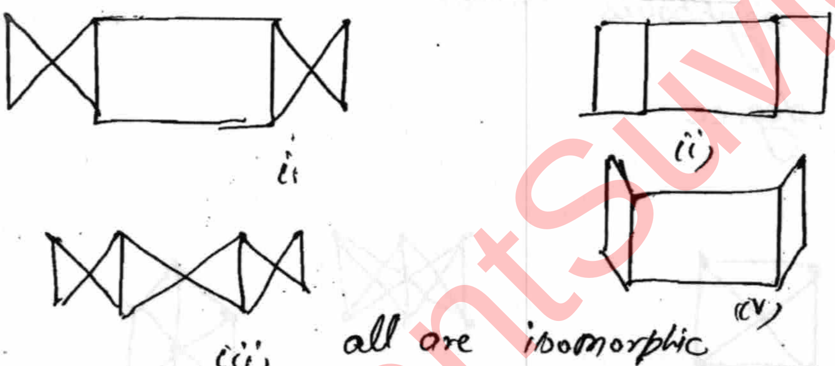
Isomorphic



Isomorphic



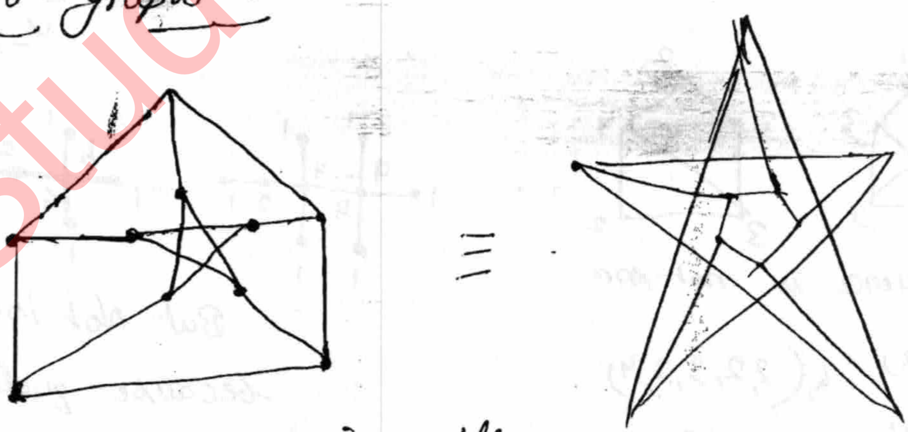
Isomorphic



all are isomorphic

Peterson graph :-

It is 3 Regular graph



isomorphic

$A = \{1, 2, 3, 4, 5\}$

$S = \{$ set of two element subset $5C_2\}$