

**B.E.**  
**Third Semester Examination, Dec-2008**  
**MATHEMATICS-III**

Note : Attempt any five questions, selecting at least one question from each part.

**Part-A**

Q. 1. (a) Expand  $f(x) = x \sin x$ ,  $0 < x < 2\pi$  as a Fourier series.

Ans.  $f(x) = x \sin x \quad 0 < x < 2\pi$

Now,  $f(x) = x \sin x$

$$f(-x) = -x \sin(-x) \\ = x \sin x$$

Means its even function,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \frac{\cos n\pi x}{C}$$

$$a_0 = \frac{2}{2\pi} \int_0^{2\pi} x \sin x dx$$

$$= \frac{1}{\pi} \left[ x(-\cos x) - \int (-\cos x) dx \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[ -x \cos x + \sin x \right]_0^{2\pi} = -\frac{1}{\pi}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} x \sin x \cos \frac{n\pi x}{2} dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x \left[ \sin \left( \frac{n\pi x}{2} + 1 \right) - \sin \left( \frac{n\pi x}{2} - x \right) \right] dx$$

$$= \frac{1}{2\pi} \left[ x \left\{ \frac{1}{n/2+1} \left[ -\cos \left( \frac{n}{2} + 1 \right) \right] + \frac{1}{n/2-1} \left( \frac{n}{2} - 1 \right) x \right\} \right]$$

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$$\begin{aligned}
 & -f\left[\frac{1}{n/2+1}\left[-\cos\left(\frac{n}{2}+1\right)\right]+\frac{1}{n/2-1}\cos\left(\frac{n}{2}-1\right)x\right]dx\Bigg]_{0}^{2\pi} \\
 & = \frac{1}{2\pi}\left[x\left\{\frac{\cos(n/2+1)x}{n/2+1}+\frac{\cos(n/2-1)x}{n/2-1}\right\}+\frac{\sin(n/2+1)x}{(n/2+1)^2}-\frac{\sin(n/2-1)x}{(n/2-1)^2}\right]_{0}^{2\pi} \\
 & = \frac{1}{2\pi}\left[2\pi\left\{\frac{\cos(n/2-1)2\pi}{(n/2-1)}-\frac{\cos(n/2+1)2\pi}{(n/2+1)}\right\}+\frac{\sin(n/2+1)2\pi}{(n/2+1)^2}-\frac{\sin(n/2-1)2\pi}{(n/2-1)^2}\right] \\
 & f(x) = -\frac{1}{\pi} + \frac{1}{2\pi}\left[(-2\pi+2) + \left(\frac{16\pi}{3}\right) + \dots\right].
 \end{aligned}$$

Q. 1. (b) Obtain a half range series for

$$\begin{aligned}
 f(x) &= kx \text{ for } 0 \leq x \leq \frac{\ell}{2} \\
 &= k(\ell-x) \text{ for } \frac{\ell}{2} \leq x \leq \ell
 \end{aligned}$$

Deduce the sum of the series

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Ans.

$$f(x) = kx \text{ for } 0 \leq x \leq 1/2$$

$$a_1 = \frac{4}{1} \int_0^{1/2} f(x) dx = \frac{4}{1} \int_0^{1/2} kx dx$$

$$= \frac{4}{1} \left[ \frac{kx^2}{2} \right]_0^{1/2}$$

$$= \frac{2}{1} \left[ \frac{kx^2}{2} \right] = \frac{k}{2}$$

$$a_2 = \frac{2}{1} \int_{1/2}^1 k(1-x) dx$$

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$$\begin{aligned} &= \frac{2k}{1} \left[ 1x - \frac{x^2}{2} \right]_{1/2} \\ &= \frac{2k}{1} \left[ 1.1 - \frac{1^2}{2} - \frac{1.1}{2} + \frac{1^2}{8} \right] \\ &= \frac{2k}{1} \left[ \frac{1^2}{8} \right] = \frac{k1}{4} \end{aligned}$$

$$\begin{aligned} f(x) &= \frac{k1}{2} + \frac{k1}{4} \\ &= \frac{k}{2} [1 + 2 + \dots] + \frac{k}{4} [1 + 2 + \dots] \end{aligned}$$

$$\begin{aligned} f(x) &= \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \\ &= \frac{1}{(2n+1)^2} \end{aligned}$$

**Q. 2. (a) State and prove convolution theorem for Fourier transforms.**

**Ans.** The convolution of two functions  $f(x)$  and  $g(x)$  is defined as,

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(u)g(x-u) dx$$

Convolution theorem on Fourier transform. The Fourier transform of the convolution of  $f(x)$  and  $g(x)$  is the product of their Fourier transform. i.e.,

$$F[f(x) * g(x)] = F[f(x)] \cdot F[g(x)]$$

**Proof :** We know that

$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) \cdot g(x-u) du$$

Taking Fourier transform of both sides of (1). We have

$$F[f(x) * g(x)] = F \left[ \int_{-\infty}^{\infty} f(u) \cdot g(x-u) \cdot du \right]$$

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$$\begin{aligned} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) \cdot g(x-u) \, du \right] e^{isx} \, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) \, du \cdot \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{\infty} g(x-u) e^{isx} \, dx \right\} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \{f(u) \cdot du \cdot Fg(x-u)\} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) \, du \cdot e^{ius} G(s) \quad (\text{using shifting property}) \\ &= G(s) \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{ius} \, du \\ &= G(s) \cdot F(s) \end{aligned}$$

By inversion

$$F^{-1}\{F(s) \cdot G(s)\} = f * g = F^{-1}\{F(s) * F^{-1}\{G(s)\}\}.$$

Q. 2. (b) Find the Fourier sine transform of

$$\frac{1}{x(x^2 + a^2)}.$$

Ans. Fourier sine transform of

$$F(x) = \frac{1}{x(x^2 + a^2)}$$

$$F_s\{f(x)\} = \int_0^{\infty} f(x) \sin sx \, dx$$

$$= \int_0^{\infty} \frac{1}{x(x^2 + a^2)} \sin sx \, dx = F(s)$$

Differentiating both sides w.r.t. S, we get

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$$\begin{aligned}\frac{d}{da}\{F(s)\} &= \int_0^{\infty} \frac{x \cos sx}{x(x^2+a^2)} \\ &= \int_0^{\infty} \frac{\cos sx}{(x^2+a^2)} = \frac{a}{s^2+a^2}\end{aligned}$$

Integration w.r.t. S, we obtain,

$$\begin{aligned}F(s) &= \int \frac{a}{s^2+a^2} ds \\ &= \tan^{-1} \frac{s}{a} + c\end{aligned}$$

But  $F(s) = 0$ , when  $s = 0$

$$\therefore c = 0$$

Hence, 
$$F(s) = \tan^{-1}\left(\frac{s}{a}\right).$$

### Part-B

Q. 3. (a) If  $\cosh x = \sec \theta$ , prove that

$$\tanh^2 \frac{x}{2} = \tan^2 \frac{\theta}{2}.$$

Ans. If  $\cosh x = \sec \theta$

Prove that  $\tan^2 \frac{x}{2} = \tan^2 \frac{\theta}{2}$

We have  $e^u = \tan\left(\frac{\pi}{4} + \frac{P}{2}\right)$

Or 
$$\frac{e^{u/2}}{e^{-u/2}} = \frac{1 + \tan \theta / 2}{1 - \tan \theta / 2}$$

By components & dividendo, we get

$$\frac{e^{u/2} - e^{-u/2}}{e^{u/2} + e^{-u/2}} = \tan \frac{\theta}{2}$$

i.e., 
$$\tanh \frac{u}{2} = \tan \frac{\theta}{2}$$

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$$\frac{1}{i} \tan \frac{i\theta}{2} = \frac{1}{i} \tanh \frac{i\theta}{2}$$

Or 
$$\frac{i\theta}{2} = \tanh^{-1} \left( \tan \frac{i\theta}{2} \right)$$
$$= \frac{1}{2} \log \frac{1 + \tan i\theta/2}{1 - \tan i\theta/2}$$

By taking Antilog both sides

$$\tan^2 \frac{x}{2} = \tan^2 \frac{\theta}{2}$$

Q. 3. (b) Reduce  $\tan^{-1}(\cos\theta + i\sin\theta)$  to the form  $x+iy$ . Hence show that

$$\tan^{-1}(e^{i\theta}) = \frac{\pi\pi}{2} + \frac{\pi}{4} - \frac{i}{2} \log \tan \left( \frac{\pi}{2} - \frac{\theta}{2} \right).$$

Ans. 
$$\tan^{-1}(\cos\theta + i\sin\theta)$$

Let 
$$\tan^{-1}(\cos\theta + i\sin\theta) = x + iy$$

Then 
$$\cos\theta + i\sin\theta = \tan(x + iy)$$
$$= \tan x \sec hy + i \sec x \tanh y$$

$$\therefore \cos\theta = \tan x \sec hy \quad \dots(1)$$

& 
$$\sin\theta = \sec x \tanh y \quad \dots(2)$$

Squaring & adding equation (1) & (2)

$$1 = \tan^2 x (\sec hy)^2 + \sec^2 x (\tanh y)^2$$
$$= \sin^2 x + \sinh^2 y (\sin^2 x + \cos^2 x)$$

$$1 - \sin^2 x = \sinh^2 y$$

i.e., 
$$\cos^2 x = \sinh^2 y$$

Now, 
$$\tan^{-1}(e^{i\theta}) = \tan \sec hy + \sec x \tanh y$$

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$$\tan^{-1}(e^{-i\theta}) = \frac{n\pi}{2} + \frac{\pi}{4} - \frac{i}{2} \log\left(\frac{\pi + i\theta}{2}\right).$$

Q. 4. (a) State and prove C-R equations and show that these are necessary for a function to be analytic in a region.

If real part of an analytic function is

$$x^3 - 3xy^2 + 3x^2 - 3y^2 + 1,$$

find the function  $f(z)$ .

Ans. **Theorem :** The necessary and sufficient conditions for the derivative of the  $w = u(x, y) + iv(x, y) = f(z)$  to exist for all values of  $z$  in a region  $R$ , are

- (i)  $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$  are continuous functions of  $x$  and  $y$  in  $R$ ;
- (ii)  $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

The relations (ii) are known as Cauchy-Riemann equations or briefly C-R equation

(a) Condition is necessary :

Let  $\delta u$  and  $\delta v$  be the increments of  $u$  and  $v$  respectively corresponding to the increments  $\delta x$  and  $\delta y$  of  $x$  and  $y$ , so that  $\delta z = \delta x + i\delta y$ .

If  $f(z)$  possesses a unique derivative at  $P(z)$ , then

$$f'(z) = \lim_{\delta z \rightarrow 0} \frac{f(z + \delta z) - f(z)}{\delta z} = \lim_{\delta z \rightarrow 0} \frac{(u + \delta u) + i(v + \delta v) - (u + iv)}{\delta z} = \lim_{\delta z \rightarrow 0} \left( \frac{\delta u}{\delta z} + i \frac{\delta v}{\delta z} \right)$$

Since  $\delta z$  can approach zero in any manner, we can first assume  $\delta z$  to be wholly real wholly imaginary. When  $\delta z$  is wholly real, then  $\delta y = 0$  and  $\delta z = \delta x$ .

$$f'(z) = \lim_{\delta x \rightarrow 0} \left( \frac{\delta u}{\delta x} + i \frac{\delta v}{\delta x} \right) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

When  $\delta z$  is wholly imaginary, then  $\delta x = 0$  and  $\delta z = i\delta y$ .

$$f'(z) = \lim_{\delta y \rightarrow 0} \left( \frac{\delta u}{i\delta y} + i \frac{\delta v}{i\delta y} \right) = \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} - i \frac{\partial v}{\partial y}$$

Now the existence of  $f'(z)$  requires the equality of (1) and (2).

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} - i \frac{\partial v}{\partial y} \quad \dots (2)$$

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On equating the real and imaginary parts from both sides, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \dots(3)$$

Thus, the necessary conditions for the existence of the derivative of  $f(z)$  is that the C-R equations should be satisfied.

**(b) Condition is sufficient :**

Suppose  $f(z)$  is a single-valued function possessing partial derivatives  $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$  at each point of the region and the C-R equations (3) are satisfied,

Then by Taylor's theorem for functions of two variables (p. 176).

$$\begin{aligned} f(z + \delta z) &= u(x + \delta x, y) + i v(x + \delta x, y + \delta y) \\ &= u(x, y) + \left( \frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y \right) + \dots + i \left( v(x, y) + \left( \frac{\partial v}{\partial x} \delta x + \frac{\partial v}{\partial y} \delta y \right) + \dots \right) \\ &= f(z) + \left( \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \delta x + \left( \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) \delta y \end{aligned}$$

[Omitting terms beyond the first powers of  $\delta x$  and  $\delta y$ ]

$$f(z + \delta z) - f(z) = \left( \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \delta x + \left( \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) \delta y$$

Now using the C-R equations (3), replace  $\frac{\partial u}{\partial y}$  and  $\frac{\partial v}{\partial y}$  by  $-\frac{\partial v}{\partial x}$  and  $\frac{\partial u}{\partial x}$  respectively.

Then

$$\begin{aligned} f(z + \delta z) &= \left[ \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right] \delta x + \left[ -\frac{\partial v}{\partial x} + i \frac{\partial u}{\partial x} \right] \delta y = \left[ \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right] \delta x + \left[ i \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} \right] i \delta y \\ &= \left[ \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right] (\delta x + i \delta y) = \left[ \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right] \delta z \end{aligned}$$

$$f'(z) = \lim_{\delta z \rightarrow 0} \frac{f(z + \delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \text{ or } \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

Real part of analytic function,

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

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Let  $f(z) = u + iv$

Where,  $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \\ &= \frac{\partial u}{\partial x} - i \frac{\partial v}{\partial y} \\ &= 3x^2 - 3y^2 + 6x - i(-6y - 6xy) \\ &= 3x^2 - 3y^2 + 6x + i(6y + 6xy) \end{aligned}$$

By Milne-Thomson's method, we express  $f(z)$  in terms of  $z$  by putting  $x = z$  &  $y = 0$

$$\begin{aligned} \therefore f'(z) &= 3z^2 + 6z + i(0) \\ &= 3z^2 + 6z \end{aligned}$$

Integrating w.r.t  $z$ , we get

$$\begin{aligned} f(z) &= \frac{3z^3}{3} + \frac{6z^2}{2} \\ &= z^3 + 3z^2 + ic \\ f(z) &= (z+3)z^2 + ic \end{aligned}$$

Q. 4. (b) Evaluate  $\int_0^{2\pi} \frac{d\theta}{1 - 2a \sin \theta + a^2}$ ,  $\theta < a < 1$ .

Ans.  $I = \int_0^{2\pi} \frac{d\theta}{1 - 2a \sin \theta + a^2}$

Putting  $z = e^{i\theta}$ ,  $d\theta = dz/iz$ ,

$$\cos \theta = \frac{1}{2} \left( z + \frac{1}{z} \right)$$

And  $\sin \theta = \frac{1}{2} \left( z - \frac{1}{z} \right)$

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$$I = \int_C \frac{1}{1 - 2a \frac{1}{2} \left( z - \frac{1}{z} \right) + a^2} \frac{dz}{iz} = \frac{1}{i} \int_C \frac{z}{z - az^2 + a + a^2 iz^2} dz$$

$$= \frac{1}{i} \int_C f(z) dz \text{ where } C \text{ is the unit circle}$$

Now  $f(z)$  has simple poles at  $z = a, \frac{1}{a}$  and the second order pole at  $z = 0$  of which the poles at  $z = 0$  and  $z = a$  lie within the unit circle,

$$\begin{aligned} \text{Res } f(a) &= \lim_{z \rightarrow a} \{ (z-a) f(z) \} \\ &= \frac{1}{i} \left[ \frac{z}{z - az^2 + a + a^2 iz^2} \right] \\ &= \frac{1}{i} \frac{a}{(a - a^3 + a + a^4 i)} \end{aligned}$$

&

$$\begin{aligned} \text{Res } f(0) &= \lim_{z \rightarrow 0} \frac{d}{dz} \{ z^2 f(z) \} \\ &= \frac{1}{i} \lim_{z \rightarrow 0} \frac{d}{dz} \frac{z}{z - az^2 + a + a^2 iz^2} \\ &= -\frac{1+a^2}{2ia^2} \end{aligned}$$

Hence,

$$\begin{aligned} I &= 2\pi i \{ \text{Res } f(a) + \text{Res } f(0) \} \\ &= 2\pi i \left[ \frac{a}{i(a - a^3 + a + a^4 i)} - \frac{1+a^2}{2ia^2} \right] \end{aligned}$$

Q. 5. (a) State and prove Residue theorem and use it to evaluate  $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^3(z-2)} dz$

Where  $C$  is the circle  $|z| = 3$

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**Ans. Singular points of an analytic function :**

We have already defined a singular point of a function as the point at which the function ceases to be analytic. If  $z = a$  is such a singular point of the function  $f(z)$  then there exists a circle with centre  $a$  which has no other singular point  $f(z)$ , then  $z = a$  is called as isolated singular point. In such a case,  $f(z)$  can be expanded in a Laurent's series around  $z = a$ , giving

$$f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + \dots + c_{-1}(z-a)^{-1} + c_{-2}(z-a)^{-2} + \dots \quad \dots(1)$$

If all the negative powers of  $(z-a)$  in (1) after the  $n$ th are missing, then the singular point  $z = a$  is called a pole of order  $n$ . A pole of first order is called a simple pole.

If the number of negative powers of  $(z-a)$  in (1) is infinite; then  $z = a$  is called as essential singularity.

2. Residues : The co-efficient of  $(z-a)^{-1}$  in the expansion of  $f(z)$  around an isolated singularity is called the residue of  $f(z)$  at that point. Thus from (1), the residue  $f(z)$  at  $z = a$  is  $c_{-1}$ .

$$\therefore \text{Res}f(a) = \frac{1}{2\pi i} \int_C f(z) dz$$

$$\text{i.e.,} \quad \int_C f(z) dz = 2\pi i \text{Res}f(a)$$

**Residue theorem :** If  $f(z)$  is analytic in a closed curve  $C$  except at a finite number of singular points within  $C$ , then

$$\int_C f(z) dz = 2\pi i \times (\text{sum of the residues at the singular points within } C)$$



Let us surround each of the singular points  $a_1, a_2, \dots, a_n$  by a small circle such that it encloses no other singular point. Then these circles  $C_1, C_2, \dots, C_n$  together with  $C$ , form a multiply connected region in which  $f(z)$  is analytic.

$\therefore$  Applying Cauchy's theorem, we have

$$\int_C f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz + \dots + \int_{C_n} f(z) dz$$

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$$= 2\pi i [\text{Res}f(a_1) + \text{Res}f(a_2) + \dots + \text{Res}f(a_n)]$$

which is the desired result.

Evaluate  $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)} dz$ , where  $C$  is the circle  $|z|=3$ .

$$f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)}$$

Analytic within the circle  $|z|=3$  excepting the poles  $z=1$  and  $z=2$ .

Since  $z=1$  is a pole of order 2.

$$\begin{aligned} \therefore \text{Res}f(1) &= \frac{1}{1!} \left[ \frac{d}{dz} \left\{ (z-1)^2 f(z) \right\} \right]_{z=1} = \left[ \frac{d}{dz} \left( \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} \right) \right]_{z=1} \\ &= \left[ \frac{(z-2)(2\pi z \cos \pi z^2 - 2\pi z \sin \pi z^2) - (\sin \pi z^2 + \cos \pi z^2)}{(z-2)^2} \right]_{z=1} \\ &= (-1)(-2\pi) - (-1) = 2\pi + 1 \end{aligned}$$

Also 
$$\text{Res}f(2) = \lim_{z \rightarrow 2} \{(z-2) f(z)\} = \lim_{z \rightarrow 2} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2} = 1$$

Hence by residue theorem,

$$\int_C f(z) dz = 2\pi i [\text{Res}f(1) + \text{Res}f(2)] = 2\pi i (2\pi + 1 + 1) = 4\pi(\pi + 1)i$$

Q. 5. (b) Expand  $\frac{z^2 - 1}{(z+2)(z+3)}$  for  $|z|=3$

What is the difference between Taylor's series and Laurent's series of a function?

Ans. Expand  $\frac{z^2 - 1}{(z+1)(z+3)}$  for  $|z|=1$

By partial fraction

$$\frac{z^2 - 1}{(z+2)(z+3)} = \frac{z^2}{(z+2)(z+3)} - \frac{1}{(z+2)(z+3)}$$

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$$= z^2(z+2)^{-1}(z+3)^{-1} - (z+2)^{-1}(z+3)^{-1} \quad \dots(2)$$

For  $|z| < 1$  both  $|z+2|$  and  $|z+3|$  are less than 1. Hence equation (2) is given as

$$\begin{aligned} f(z) &= -\frac{1}{2} \left( 1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \dots \right) \\ &\quad + \left( 1 + z + z^2 + z^3 + \dots \right) \\ &= \frac{1}{2} + \frac{3}{4}z + \frac{7}{8}z^2 + \frac{15}{16}z^3 + \dots \end{aligned}$$

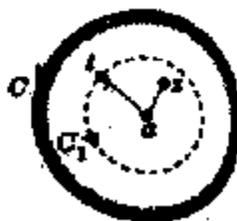
Which is a Taylor's series.

If  $f(z)$  is analytic inside a circle with  $C$  with centre at  $a$ , then for  $z$  inside  $C$ ,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \dots \quad \dots(i)$$

**Proof:** Let  $z$  be any point inside  $C$ . Draw a circle  $C_1$  with centre at  $a$  enclosing  $z$  (Fig. 18.19). Let  $t$  be a point on  $C_1$ . We have

$$\begin{aligned} \frac{1}{t-z} &= \frac{1}{t-a-(z-a)} = \frac{1}{t-a} \left( 1 - \frac{z-a}{t-a} \right)^{-1} \\ &= \frac{1}{t-a} \left[ 1 + \frac{z-a}{t-a} + \left( \frac{z-a}{t-a} \right)^2 + \dots + \left( \frac{z-a}{t-a} \right)^n + \dots \right] \quad \dots(ii) \end{aligned}$$



As  $|z-a| < |t-a|$ , i.e.,  $|(z-a)/(t-a)| < 1$ , this series converges uniformly. So, multiplying both sides of (ii) by  $f(t)$ , we can integrate over  $C_1$ .

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$$\int_{C_1} \frac{f(t)}{t-z} dz = \int_{C_1} \frac{f(t)}{t-a} dt + (z-a) \int_{C_1} \frac{f(t)}{(t-a)^2} dt + \dots + (z-a)^n \int_{C_1} \frac{f(t)}{(t-a)^{n+1}} dt + \dots \quad \dots(iii)$$

Since  $f(t)$  is analytic on and inside  $C_1$ , therefore, applying the formulae (2) to (5) of p. 509 to (iii), we get (i) (i) which is known as Taylor's series.

Obs. Another remarkable fact is that complex analytic functions can always be represented by power series of the form (i).

**(iii) Laurent's series :**

If  $f(z)$  is analytic in the ring-shaped region  $R$  bounded by two concentric circles  $C$  and  $C_1$  of radii  $r$  and  $r_1$  ( $r > r_1$ ) and with centre at  $a$ , then for all  $z$  in  $R$

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \dots$$

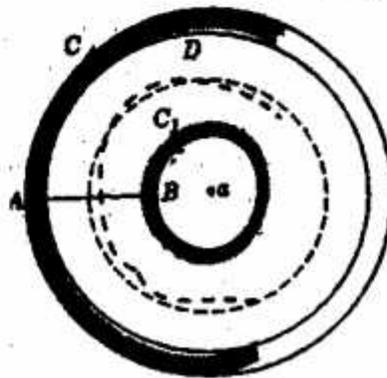
Where 
$$a_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{(t-a)^{n+1}} dt,$$

$\Gamma$  being any curve in  $R$ , encircling  $C_1$  (as in fig.)

**Proof :** Introduce cross-cut  $AB$ , then  $f(z)$  is analytic in the region  $D$  bounded by  $AB$ ,  $C_1$  described clockwise,  $BA$  and  $C$  described anti-clockwise (see fig.). Then if  $z$  be any point in  $D$ , we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \left[ \int_{AB} \frac{f(t)}{t-z} dt + \int_{C_1} \frac{f(t)}{t-z} dt + \int_{BA} \frac{f(t)}{t-z} dt + \int_C \frac{f(t)}{t-z} dt \right] \\ &= \frac{1}{2\pi i} \left[ \int_C \frac{f(t)}{t-z} dt - \int_{C_1} \frac{f(t)}{t-z} dt \right] \quad \dots(i) \end{aligned}$$

Where both  $C$  and  $C_1$  are described anti-clockwise in (i) and integrals along  $AB$  and  $BA$  cancel.



For the first integral in (i), expanding  $1/(t-z)$  as in (2) we get

$$\begin{aligned} \frac{1}{2\pi i} \int_C \frac{f(t)}{t-z} dt &= \sum_{n=1}^{\infty} \frac{(z-a)^n}{2\pi i} \int_C \frac{f(t)}{(t-a)^{n+1}} dt \\ &= \sum a_n (z-a)^n \text{ where } a_n = \frac{1}{2\pi i} \int_C \frac{f(t)}{(t-a)^{n+1}} dt \end{aligned}$$

For the second integral in (i), let  $t$  lie on  $C_1$ . Then we write

### Part-C

**Q. 6. (a)** There are three bags : first containing 1 white, 2 red, 3 green balls; second 2 white, 3 red, 1 green balls and third 3 white, 1 red, 2 green balls. Two balls are drawn from a bag chosen at random. They are found to be 1 red and 1 white. Find the probability that ball so drawn came from the second bag.

**Ans. Bag 1**

- Number of white Ball = 1
- Number of Red Ball = 2
- Number of Green Balls = 3

**Bag 2**

- Number of white Ball = 2
- Number of Red Ball = 3
- Number of Green Balls = 1

**Bag 3**

- Number of white Ball = 3
- Number of Red Ball = 1
- Number of Green Balls = 2

Probability of two ball drawn from the bag

$$\begin{aligned} &= \frac{2!}{6!4!} \\ &= \frac{1}{720 \times 6} \end{aligned}$$

Probability of 1 red ball =  $\frac{2!}{1!} = 2$

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Probability of 1 white ball = 1

Probability of getting ball from second bag

$$= \frac{2!}{6!2!}$$

$$= \frac{1}{6!}$$

Q. 6. (b) Is the function defined as follows a density function?

$$f(x) = e^{-x}, x \geq 0$$

$$= 0 \text{ otherwise}$$

If so find  $P[1 \leq X \leq 2]$ .

Ans. (i) Is the function defined as follows a density function

$$f(x) = e^{-x}, x \geq 0$$

$$= 0, x < 0$$

(i) If so, determine the probability that the variate having this density will fall in the interval (1, 2)?

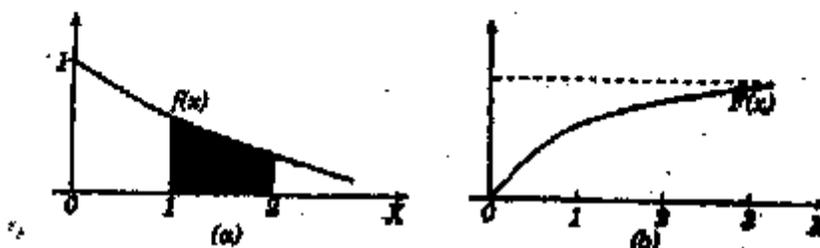
(ii) Also find the cumulative probability function  $F(x)$ ?

(i)  $f(x)$  is clearly  $\geq 0$  for every  $x$  in  $(1, 2)$  and

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^0 0 dx + \int_0^{\infty} e^{-x} dx = 1$$

Hence the function  $f(x)$  satisfies the requirements for a density function.

(ii) Required probability =  $P(1 \leq x \leq 2)$



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$$= \int_1^2 e^{-x} dx = e^{-1} - e^{-2} = 0.368 - 0.135 = 0.233$$

This probability is equal to the shaded area in fig. (a).

(iii) Cumulative probability function  $F(2)$

$$\begin{aligned} \int_{-\infty}^2 f(x) dx &= \int_{-\infty}^0 0 dx + \int_0^2 e^{-x} dx \\ &= 1 - e^{-2} = 1 - 0.135 = 0.865 \end{aligned}$$

Which is shown in fig. (b).

**Q. 7. (a) Define Poisson distribution and discuss some of its properties.**

**Ans. 1. Poisson distribution :**

It is a distribution related to the probabilities of events which are extremely rare, but which have a large number of independent opportunities for occurrence. The number of persons born blind per year in a large city and the number of deaths by horse kick in an army corps are some of the phenomena, in which this law is followed.

This distribution can be derived as a limiting case of the binomial distribution by making  $n$  very large and  $p$  very small, keeping  $np$  fixed ( $=m$ , say).

The probability of  $r$  successes in a binomial-distribution is

$$\begin{aligned} P(r) &= {}^n C_r p^r q^{n-r} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!} p^r q^{n-r} \\ &= \frac{np(np-p)(np-2p)\dots(np-r+1p)}{r!} (1-p)^{n-r} \end{aligned}$$

As  $n \rightarrow \infty$ ,  $p \rightarrow 0$  ( $np = m$ ), we have

$$P(r) = \frac{m^r}{r!} \lim_{n \rightarrow \infty} \frac{(1 - m/n)^n}{(1 - m/n)^r} = \frac{m^r}{r!} e^{-m}$$

So that the probabilities of 0, 1, 2, ...,  $r$ , ... successes in a Poisson distribution are given by

$$e^{-m}, me^{-m}, \frac{m^2 e^{-m}}{2!}, \dots, \frac{m^r e^{-m}}{r!}, \dots$$

The sum of these probabilities is unity as it should be.

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**2. Constants of the Poisson distribution :**

These constants can easily be derived from the corresponding constants of the binomial distribution simply by making  $n \rightarrow \infty$ ,  $p \rightarrow 0$ , ( $q \rightarrow 1$ ) and noting that  $np = m$

$$\text{Mean} = Lt(np) = m$$

$$\mu_2 = Lt(npq) = mLt(q) = m$$

$$\therefore \text{Standard deviation} = \sqrt{m}$$

$$\text{Also } \mu_3 = m, \quad \mu_4 = m + 3m^2$$

$$\therefore \text{Skewness } (= \sqrt{\beta_1}) = 1/m, \text{ Kurtosis } (= \beta_2) = 3 + 1/m.$$

Since  $\mu_3$  is positive, Poisson distribution is positively skewed and since  $\beta_2 > 3$ , it is Leptokurtic.

**(iii) Applications of Poisson distribution :**

This distribution is applied to product concerning : (i) Arrival pattern of 'defective vehicles in a workshop', patients in a hospital's 'telephone calls.'

(ii) Demand pattern for certain spare parts.

(iii) Number of fragments from a shell hitting a target.

(iv) Spatial distribution of bomb hits.

**Q. 7. (b) Fit a normal curve to the following distribution**

x:	2	4	6	8	10
f:	1	4	6	4	1

Ans.

x:	2	4	6	8	10
f:	1	4	6	4	1

$$\text{Mean} = \frac{\sum f_i x_i}{\sum f_i} = \frac{2 + 16 + 36 + 32 + 10}{16}$$

$$= \frac{96}{16}$$

$$= 6$$

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∴ Mean of Poisson distribution

i.e.,  $m = 6$

Hence, the theoretical frequency for  $r$  successes is

$$\frac{Ne^{-m}(m)^r}{r!} = \frac{16e^{-6}(6)^r}{r!}$$

Where  $r = 0, 1, 2, 3, 4$

∴ The theoretical frequencies are :

x:	2	4	6	8	10
f:	1	4	6	4	1

$$(\because e^{-6} = 0.61)$$

**Q. 8. (a) Using Simplex method,**

Maximize  $Z = 5x_1 + 3x_2$

subject to  $x_1 + x_2 \leq 2$ ,

$$5x_1 + 2x_2 \leq 10$$

$$3x_1 + 8x_2 \leq 12; x_1, x_2 \geq 0.$$

**Ans.**

Using simplex method

Minimize  $Z = 5x_1 + 3x_2$

Subject to  $x_1 + x_2 \leq 2, 5x_1 + 2x_2 \leq 10, 3x_1 + 8x_2 \leq 12, x_1, x_2 \geq 0$

Solution consists of the following steps :

**Step 1 :** Check whether the objective function is to be maximized and all b's are positive. The problem being of maximization type and all b's being  $\geq 0$ , this step is not necessary.

**Step 2 :** Express the problem in the standard form.

By introducing the slack variables  $s_1, s_2, s_3$ , the problem in standard form becomes

$$\text{Max. } Z = 5x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3$$

$$\text{Subject to } x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 2$$

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$$5x_1 + 2x_2 + 0s_1 + s_2 + 0s_3 = 10$$

$$3x_1 + 8x_2 + 0s_1 + 0s_2 + 0s_3 = 10$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

**Step 3 : Find an initial basic feasible solution**

There are three equations involving five unknowns and for obtaining a solution, we assign zero values to any two of the variables. We start with a basic solution for which we set  $x_1 = 0$  and  $x_2 = 0$ . (This basic solution corresponds to the origin in the graphical method). Substituting  $x_1 = x_2 = 0$  in (i), (ii) and (iii), we get the basic solution

$$s_1 = 2, s_2 = 10, s_3 = 12$$

Since all  $s_1, s_2, s_3$  are positive, the basic solution is also feasible and non-degenerate.

∴ The basic feasible solution is given by the following table :

$c_j$		5	3	0	0	0		
$c_B$ Basic	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	b		0
0	$s_1$	(1)	1	1	0	0	2	2/1 ←
0	$s_2$	5	2	0	1	0	10	10/5
0	$s_3$	3	8	0	0	1	12	12/3
	$Z_j = \sum c_B a_{ij}$	0	0	0	0	0	0	
	$C_j = c_j - Z_j$	5	3	0	0	0		
		↑						

[For  $x_1$  - column ( $j = 1$ ),  $Z_j = \sum c_B a_{i1} = 0(1) + 0(5) + 0(3) = 0$

And for  $x_2$  - column ( $j = 2$ ),  $Z_j = \sum c_B a_{i2} = 0(1) + 0(2) + 0(8) = 0$ .

Similarly  $Z_j = 0(2) + 0(10) + 0(12) = 0$ .

**Step 4 : Apply optimality test.**

As  $C_j$  is positive under some columns, the initial basic feasible solution is not optimal (i.e., can be FOR MORE STUDY MATERIAL LOG ON TO <http://studentsuvidha.in>

improved) and we proceed to the next step.

**Step 5 :** (i) Identify the incoming and outgoing variables.

The above table shows that  $x_1$  is the incoming variable as its incremental contributions  $C_j (= 5)$  is maximum and the column in which it appears is the key column (shown marked by an arrow at the bottom).

Dividing the elements under b-column by the corresponding elements of key-column, we find minimum positive ratio  $\theta$  is 2 in two rows. We, therefore, arbitrarily choose the row containing  $s_1$  as the key row (shown marked by an arrow on its right end). The element at the intersection of key row and the key column i.e., (1), the key element.  $s_1$  is therefore, the outgoing basic variable which will now become non-basic.

Having decided that  $x_1$  is to enter the solution, we have tried to find as to what maximum value  $x_1$  could have without violating the constraints. So, removing  $s_1$ , the new basic will contains  $x_1, s_2$  and  $s_3$  as the basic variables.

**(ii) Iterate towards the optimal solution :**

To transform the initial set of equations with a basic feasible solution into an equivalent set equations with a different basic feasible solution, we make the key element unity. Here the key element being unity, we retain the key row as it is. Then to make all other elements in key column zero, we subtract proper multiples of key row from the other rows. Here we subtract 5 times the elements of key row from the second row and 3 times the elements of key row from the ... row. These become the second and the third rows of the next table. We also change the corresponding value under  $c_B$  column from 0 to 5, while replacing  $s_1$  by  $x_1$  under the basis. Thus, the second basic feasible solution is given by the following table :

$c_j$		5	3	0	0	0		
$c_B$	Basic	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	b	$\theta$
5	$x_1$	1	1	1	1	0	0	2
0	$s_2$	0	-3	-5	1	0	0	0
0	$s_3$	0	5	-3	0	0	1	6
	$Z_j = \sum c_B a_{ij}$	5	5	5	0	0	0	10
	$C_j = c_j - Z_j$	0	-2	-5	0	0		

As  $C_j$  is either zero or negative under all columns, the above table gives the optimal basic table solution.

This optimal solution is  $x_1 = 2, x_2 = 0$  and maximum  $Z = 10$ .

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**Q. 8. (b) Using dual Simplex method**

**Maximize**  $z_r = -3x_1 - x_2$

**Subject to**  $x_1 + x_2 \geq 1,$

$2x_1 + 3x_2 \geq 2;$

$x_1, x_2 \geq 0.$

**Ans. Consists of the following steps :**

**Step 1 :** (i) Convert the first and third constraints into ( $\leq$ ) type. These constraints become

$$-x_1 - x_2 \leq -1, -x_1 - 2x_2 \leq -10$$

(ii) Express the problem in standard form

Introducing slack variables  $s_1, s_2, s_3, s_4$  the given problem takes the form

**Max.**  $Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

**Subject to**  $-x_1 - x_2 + s_1 = -1, x_1 + x_2 + s_2 = 7, -x_1 - 2x_2 + s_3 = -10,$

$x_2 + s_4 = 3, x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$

**Step 2 :** Find the initial basic solution

Setting the decision variables  $x_1, x_2$  each equal to zero, we get the basic solution

$$x_1 = x_2 = 0; s_1 = -1, s_2 = 7, s_3 = -10, s_4 = 3 \text{ and } Z = 0$$

$\therefore$  Initial solution is given by the table below :

$c_j$		-3	-2	0	0	0	0	
$c_B$	Basic $x_i$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$	$b$	
0	$s_1$	-1	-1	1	0	0	0	-1
0	$s_2$	1	1	0	1	0	0	7
0	$s_3$	-1	(-2)	0	0	1	0	-10 ←
	$Z_j = \sum c_B a_{ij}$	0	0	0	0	0	0	0
	$C_j = c_j - Z_j$	-3	-2	0	0	0	0	0
			↑					

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**Step 3 :** Test nature of  $C_j$ .

Since all  $C_j$  values are  $\leq 0$  and  $b_1 = -1$ ,  $b_3 = -10$ , the initial solution is optimal but infeasible. We therefore, proceed further.

**Step 4 :** Mark the outgoing variable.

Since  $b_3$  is negative and numerically largest, the third row is the key row and  $s_3$  is the outgoing variable.

**Step 5 :** Calculate ratios of elements in  $C_j$ -row to the corresponding negative elements of the key row.

These ratios are  $-3/-1=3$ ,  $-2/-2 = 1$  (negative ratios corresponding to +ve or zero elements of key row). Since the smaller ratio is 1, therefore,  $x_2$ -column is the key column and  $(-2)$  is the key element.

**Step 6 :** Iterate towards optimal feasible solution.

(i) Drop  $s_3$  and introduce  $x_2$  alongwith its associated value  $-2$  under  $c_B$  column. Convert the key element to unity and make all other elements of the key column zero. Then the second solution is given by the table below :

	$c_j$	$-3j$	$-2$	$0$	$0$	$0$	$0$	
$c_B$	Basic	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$	$b$
$0$	$s_1$	$-\frac{1}{2}$	$0$	$1$	$0$	$0$	$-\frac{1}{2}$	$4$
$0$	$s_2$	$\frac{1}{2}$	$0$	$0$	$1$	$0$	$\frac{1}{2}$	$2$
$2$	$x_2$	$\frac{1}{2}$	$1$	$0$	$0$	$0$	$-\frac{1}{2}$	$5$
$0$	$s_4$	$\left(-\frac{1}{2}\right)$	$0$	$0$	$0$	$0$	$\frac{1}{2}$	$1$ $-2 \leftarrow$
	$Z_j = \sum c_B a_{ij}$	$-1$	$-2$	$0$	$0$	$0$	$1$	$-10$
	$C_j = c_j - Z_j$	$-2$	$0$	$0$	$0$	$0$	$-1$	$0$
			$\uparrow$					

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Since all  $C_j$  values are  $\leq 0$  and  $b_4 = -2$ , this solution is optimal but infeasible. We therefore further.

(ii) Mark the outgoing variable.

Since  $b_4$  is negative, the fourth row is the key row and  $s_4$  is the outgoing variable.

(iii) Calculate ratios of elements in  $C_j$ -row to the corresponding negative elements of the key

This ratios is  $-2 / -\frac{1}{2} = 4$  (neglecting other ratios corresponding to +ve or 0 elements of key row)

$\therefore x_1$ -column is the key column and  $\left(-\frac{1}{2}\right)$  is the key element.

(iv) Drop  $s_4$  and introduce  $x_1$  with its associated value  $-3$  under the  $c_B$  column. Convert the elements in the key row to 1 and make all other elements of the key column zero. Then the third solution given by the table below

	$C_j$	-3	-2	0	0	0	0		
$c_B$	Basic	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$	$b$	
0	$s_1$	0	0	1	0	0	-1	-1	6
0	$s_2$	0	0	0	1	1	1	1	0
2	$x_2$	0	1	0	0	0	0	1	3
3	$x_1$	1	0	0	0	0	-10	-2	4
	$Z_j$	-3	-2	0	0	0	3	4	-18