

B.E.
Third Semester Examination, May-2008
MATHEMATICS-III

Note : Attempt any five questions.

Part-A

Q. 1. (a) Obtain a Fourier series to represent e^{-ax} from $x = -\pi$ to $x = \pi$. Hence derive series for $\frac{\pi}{\sinh \pi}$.

Ans. Let
$$e^{-ax} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

According to Euler's formulae,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-ax} dx = \frac{1}{\pi} \left[\frac{e^{-ax}}{-a} \right]_{-\pi}^{\pi} = -\frac{1}{a\pi} (e^{-a\pi} - e^{a\pi}) = \frac{2 \sinh a\pi}{a\pi}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-ax} \cos nx dx = \frac{1}{\pi} \left[e^{-ax} \frac{\sin nx}{n} - \int_{-\pi}^{\pi} e^{-ax} (-a) \frac{\sin nx}{n} dx \right]$$

$$= \frac{1}{\pi} \left[\frac{e^{-ax}}{a^2 + n^2} (-a \cos nx + n \sin nx) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{e^{-a\pi}}{a^2 + n^2} (-a(1)^n) - \frac{e^{a\pi}}{a^2 + n^2} (-a(-1)^n) \right]$$

$$= \frac{(-1)^n a 2 \sinh a\pi}{(a^2 + n^2)\pi}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-ax} \sin nx dx = \frac{1}{\pi} \left[e^{-ax} \left(\frac{-\cos nx}{n} \right) - \int_{-\pi}^{\pi} e^{-ax} (-a) \left(\frac{-\cos nx}{n} \right) dx \right]$$

$$= \frac{1}{\pi} \left[\frac{e^{-ax}}{a^2 + n^2} (a \sin nx + n \cos nx) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{e^{-a\pi}}{a^2 + n^2} (n(-1)^n) - \frac{e^{-a(-\pi)}}{a^2 + n^2} (n(-1)^n) \right]$$

$$b_n = \frac{2n(-1)^n \sinh a\pi}{\pi(a^2 + n^2)}$$

Putting the value of a_0 , a_n & b_n we get,

$$e^{-ax} = \frac{2 \sinh a\pi}{\pi} \left[\frac{1}{2a} - a \left(\frac{\cos x}{a^2 + 1^2} - \frac{\cos 2x}{a^2 + 2^2} + \dots \right) - \left(\frac{\sin x}{a^2 + 1^2} - \frac{2 \sin 2x}{a^2 + 2^2} + \dots \right) \right]$$

Put $x = 0$, $a = 1$

$$1 = \frac{2 \sinh \pi}{\pi} \left[\frac{1}{2} - \left(\frac{1}{2} - \frac{1}{2^2 + 1} + \frac{1}{3^2 + 1} - \dots \right) \right]$$

\Rightarrow

$$\frac{\pi}{\sinh \pi} = 2 \left[\frac{1}{2^2 + 1} - \frac{1}{3^2 + 1} + \frac{1}{4^2 + 1} - \dots \right]$$

Q. 1. (b) Obtain a half range cosine series for

$$f(x) = \begin{cases} kx & \text{for } 0 \leq x \leq \frac{1}{2} \\ k(1-x) & \text{for } \frac{1}{2} \leq x \leq 1 \end{cases}$$

and hence find sum of the series

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Ans. $f(x) = kx \quad 0 \leq x \leq 1/2$

$= k(1-x) \quad 1/2 \leq x \leq 1$

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$f(x)$ may be consider as even function in $(-1,1)$ by extending function in $(-1, 0)$ in such a way that it become symmetrical about y-axis.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

By Euler's formula,

$$a_0 = \frac{2}{l} \int_0^l f(x) dx = \frac{2}{l} \left[\int_0^{l/2} kx dx + \int_{l/2}^l k(l-x) dx \right]$$

$$= \frac{2}{l} \left[\left(\frac{kx^2}{2} \right)_0^{l/2} + \left\{ k \left(lx - \frac{x^2}{2} \right) \right\}_{l/2}^l \right]$$

$$= \frac{2}{l} \left[\frac{kl^2}{8} + k \left(l^2 - \frac{l^2}{2} \right) - k \left(\frac{l^2}{2} - \frac{l^2}{8} \right) \right] = \frac{kl}{2}$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx = \frac{2}{l} \left[\int_0^{l/2} kx \cos \frac{n\pi x}{l} dx + \int_{l/2}^l k(l-x) \cos \frac{n\pi x}{l} dx \right]$$

$$= \frac{2}{l} \left[k \left\{ x \frac{\sin n\pi x}{n\pi/l} + \frac{\cos n\pi x}{(n\pi/l)^2} \right\}_0^{l/2} + k \left\{ (l-x) \frac{\sin n\pi x}{n\pi/l} - \frac{\cos n\pi - \cos n\pi/2}{(n\pi/l)^2} \right\} \right]$$

$$= \frac{2}{l} \left[k \left\{ \frac{l}{2} \frac{\sin n\pi/2}{n\pi/l} + \frac{\cos n\pi/2}{(n\pi/l)^2} - \frac{1}{(n\pi/2)} \right\} + k \left\{ -\frac{l}{2} \frac{\sin n\pi/2}{n\pi/l} - \frac{\cos n\pi - \cos n\pi/2}{(n\pi/l)^2} \right\} \right]$$

$$= \frac{2kl}{n^2 \pi^2} \left[2 \cos \frac{n\pi}{2} - 1 - (-1)^n \right]$$

$$a_1 = 0, a_2 = \frac{2kl}{2^2 \pi^2} (-2 - 1 - 1) = -\frac{2kl}{\pi^2}$$

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$$a_3 = 0, a_4 = 0, a_5 = 0, a_6 = \frac{2kl}{(6)^2 \pi^2} (-2 - 1 - 1) = \frac{2kl}{3^2 \pi^2}$$

$$f(x) = \frac{kl}{4} - \frac{2kl}{\pi^2} \left[\frac{\cos 2x \frac{\pi}{l}}{1^2} + \frac{\cos 6x \pi / l}{3^2} + \frac{\cos 10x \pi / l}{5^2} + \dots \right]$$

Put $x = 0$

$$f(x) = \frac{kl}{4} - \frac{2kl}{\pi^2} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

Or $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

Q. 2. (a) Express the function :

$$f(x) = \begin{cases} 1 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

as a Fourier integral

Hence evaluate $\int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda$.

Ans. $f(x) = \begin{cases} 1 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| > 1 \end{cases}$

The Fourier integral for

$$\begin{aligned} f(x) &= \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) \cos \lambda(t-x) dt d\lambda \\ &= \frac{1}{\pi} \int_0^\infty \int_{-1}^1 1 \cdot \cos \lambda(t-x) dt d\lambda = \frac{1}{\pi} \int_0^\infty \left[\frac{\sin \lambda(t-x)}{\lambda} \right]_{-1}^1 d\lambda \\ &= \frac{1}{\pi} \int_0^\infty \frac{\sin \lambda(1-x) - \sin \lambda(-1-x)}{\lambda} d\lambda \end{aligned}$$

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$$= \frac{1}{\pi} \int_0^{\infty} \frac{\sin \lambda(1-x) + \sin \lambda(1+x)}{\lambda} d\lambda = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda$$

$$\therefore \int_0^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda = \frac{1}{2} \pi f(x)$$

$$= \begin{cases} \frac{1}{2} \pi & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$

at $x = \pm 1$, the function is discontinuous & hence integral has the value $\frac{1}{2} \left(\frac{\pi}{2} + 0 \right) = \frac{\pi}{4}$.

Q. 2. (b) State and prove convolution theorem for Fourier transforms.

Ans. Fourier transform of convolution theorem :

The convolution of two function $f(x)$ & $g(x)$ over the interval $(-\infty, \infty)$ is defined as,

$$f * g = \int_{-\infty}^{\infty} f(u)g(x-u) du = h(x)$$

Proof : The Fourier transform of convolution of $f(x)$ & $g(x)$ is product of their Fourier transform i.e.,

$$F(f(x) * g(x)) = F(f(x)).F(g(x))$$

We have,

$$\begin{aligned} F\{f(x) * g(x)\} &= F\left[\int_{-\infty}^{\infty} f(u)g(x-u) du\right] \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x)g(x-u) du\right] e^{-ikx} dx = \int_{-\infty}^{\infty} f(x) \left[\int_{-\infty}^{\infty} g(x-u) \cdot e^{iux} dx\right] du \\ &= \int_{-\infty}^{\infty} f(x) \left[\int_{-\infty}^{\infty} e^{i(x-u)u} \cdot g(x-u) d(x-u)\right] e^{isu} du \\ &= \int_{-\infty}^{\infty} e^{-iku} f(x) \left[\int_{-\infty}^{\infty} e^{iut} g(t) dt\right] du \text{ when } x-u = t \\ &= \int_{-\infty}^{\infty} e^{isu} f(u) du \cdot F(g(t)) \end{aligned}$$

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$$= \int_{-\infty}^{\infty} e^{isx} f(x) dx \cdot F[g(x)]$$

$$= F[f(x)] \cdot F[g(x)].$$

Part-B

Q. 3. (a) Define analytic functions. State and prove the C-R equations for an analytic function.

Ans. Analytic function :

Consider a single valued function $f(z)$ in a domain D . The function $f(z)$ is said to be analytic at point $z = a$ if there exists a neighbourhood $|z - a| < \delta$ at all points of which $f'(z)$ exists.

If $f'(z)$ exists at every point of a domain, then we can say that the function is analytic or regular analytic in the domain.

Necessary and sufficient conditions for $f(z)$ to be analytic : The necessary and sufficient conditions for the function

$$w = f(z) = u(x, y) + iv(x, y)$$

To be analytic in a region R , are

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y in the region R .

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

The conditions in (ii) are known as Cauchy-Riemann equations or briefly C-R equations.

Proof : (a) Necessary condition : Let $f(z) = u + iv$ be a single-valued function possessing partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at each point of a region R and satisfying C-R equations.

i.e.,
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

We shall show that $f(z)$ is analytic i.e., $f'(z)$ exists at every point of the region R .

By Taylor's theorem for functions of two variables, we have, on omitting second and higher degree terms of δx and δy .

$$f(z + \delta z) = u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y)$$

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$$\begin{aligned}
 &= \left[u(x, y) + \left(\frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y \right) \right] + i \left[v(x, y) + \left(\frac{\partial v}{\partial x} \delta x + \frac{\partial v}{\partial y} \delta y \right) \right] \\
 &= [u(x, y) + i v(x, y)] + \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \delta x + \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) \delta y \\
 &= f(z) + \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \delta x + \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) \delta y
 \end{aligned}$$

$$\begin{aligned}
 \text{Or } f(z + \delta z) - f(z) &= \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \delta x + \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) \delta y \\
 &= \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \delta x + \left(-\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \delta y \quad [\text{Using C-R equations}] \\
 &= \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \delta x + \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) i \delta y \quad [\because -1 = i^2] \\
 &= \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) (\delta x + i \delta y) = \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \delta z \quad [\because \delta x + i \delta y = \delta z]
 \end{aligned}$$

$$\Rightarrow f'(z) = \lim_{\delta z \rightarrow 0} \frac{f(z + \delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

Thus $f'(z)$ exists, because $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$ exist.

Hence $f(z)$ is analytic.

Q. 3. (b) If $f(z) = u + iv$ is analytic function, find $f'(z)$, if $u - v = e^x (\cos y - \sin y)$.

Ans. $u = v = e^x (\cos y - \sin y)$

$$\therefore \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = e^x (\cos y - \sin y) \quad \dots (i)$$

$$\& \quad \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = e^x [-\sin y - \cos y]$$

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$$-\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x} = e^x [-\sin y - \cos y] \quad \dots(ii)$$

Subtracting (ii) from (i)

$$\begin{aligned} \frac{\partial u}{\partial x} &= e^x \cos y - e^x \sin y + e^x \sin y + e^x \cos y \\ &= 2e^x \cos y \end{aligned}$$

Adding (i) & (ii)

$$-\frac{2dv}{dx} = +2e^x \sin y$$

Thus,

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y}$$

and putting $x = z$ & $y = 0$

$$\begin{aligned} f'(z) &= (e^z \cos 0 + ie^z \sin 0) \\ &= e^z \\ f(z) &= \int e^z dz + c \\ &= e^z + c \end{aligned}$$

Q. 4. (a) Show that $\int_C (z+1)dz = 0$ where C is the boundary of the square whose vertices are at the points $z = 0$, $z = 1$, $z = 1+i$ and $z = i$.

Ans. $\int_C (z+1)dz$

$$\begin{aligned} &= \int_0^1 (z+1)dz + \int_1^{1+i} (z+1)dz + \int_{1+i}^i (z+1)dz + \int_i^0 (z+1)dz \\ &= \left(\frac{z^2}{2} + z \right)_0^1 + \left(\frac{z^2}{2} + z \right)_1^{1+i} + \left(\frac{z^2}{2} + z \right)_{1+i}^i + \left(\frac{z^2}{2} + z \right)_i^0 \\ &= \left(\frac{3}{2} - 0 \right) + \left[\left(\frac{(1+i)^2}{2} + (1+i) \right) - \frac{3}{2} \right] + \left[\left(\frac{i^2}{2} + i \right) - \left(\frac{(1+i)^2}{2} + (1+i) \right) \right] \end{aligned}$$

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$$+ \left[0 - \left(\frac{i^2}{2} + 1 \right) \right]$$

$$= 0 = \text{RHS.}$$

Q. 4. (b) Evaluate $\int_C \frac{z^3 + z + 1}{z^2 - 7z + 2} dz$, where C is the ellipse

$$4x^2 + 9y^2 = 1.$$

Ans. We know $z = x + iy \Rightarrow dz = dx + i dy$

First we have to check whether the function is analytic considering,

$$\begin{aligned} z^3 + z + 1 &= (x + iy)^3 + (x + iy) + 1 \\ &= (x^3 - 3xy^2 + x + i) + i(3x^2y - y^3 + y) \end{aligned}$$

Now, $\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 1; \frac{\partial u}{\partial y} = -6xy; \frac{\partial v}{\partial x} = 6xy; \frac{\partial v}{\partial y} = 3x^2 - 3y^2 + 1$

i.e. Cauchy Riemann equations are satisfied, hence the above function is analytical.

Considering $z^2 - 7z + 2 = (x + iy)^2 - 7(x + iy) + 2$

$$= (x^2 - y^2 - 7x + 2) + i(2xy - 7y)$$

$$\frac{\partial u}{\partial x} = 2x - 7; \frac{\partial v}{\partial x} = 2y; \frac{\partial u}{\partial y} = -2y; \frac{\partial v}{\partial y} = 2x - 7$$

Hence Cauchy's Riemann's equations are satisfied the function is analytic.

Therefore, the function $\frac{z^3 + z + 1}{z^2 - 7z + 2}$ is analytic everywhere with in and on $4x^2 + 9y^2 = 1$.

\therefore By Cauchy's integral theorem.

$$\oint_C f(z) dz = 0$$

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Q. 4. (e) Expand $\frac{e^z}{(z-1)^2}$ about $z = 1$.

Aus. Let $z - 1 = t$ or $z = 1 + t$

$$\begin{aligned} \therefore f(z) &= \frac{e^{(t+1)}}{t^2} = \frac{e}{t^2} \cdot e^t \\ &= \frac{e}{t^2} \left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right] \\ &= e \left[\frac{1}{t^2} + \frac{1}{t} + \frac{1}{2!} + \frac{t}{3!} + \frac{t^2}{4!} + \dots \right] \\ &= e \left[\frac{1}{(z-1)^2} + \frac{1}{(z-1)} + \frac{1}{2!} + \frac{(z-1)}{3!} + \frac{(z-1)^2}{4!} + \dots \right] \\ &= e \left[(z-1)^{-2} + (z-1)^{-1} + \frac{1}{2!} + \frac{1}{3!}(z-1) + \frac{1}{4!}(z-1)^2 + \dots \right]. \end{aligned}$$

Q. 5. (a) Evaluate

$\int_C \frac{e^z}{\cos \pi z} dz$, where C is the unit circle $|z| = 1$. How many poles the function $\frac{e^z}{\cos \pi z}$,

Aus. $f(z) = \frac{e^z}{\cos \pi z}$ has simple poles at

$$z = \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \dots$$

Of which only $z = \pm \frac{1}{2}$ lies inside the circle $|z| = 1$.

Residue of $f(z)$ at $z = \frac{1}{2}$ is

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$$\begin{aligned}\lim_{z \rightarrow \frac{1}{2}} \left(z - \frac{1}{2} \right) f(z) &= \lim_{z \rightarrow \frac{1}{2}} \frac{\left(z - \frac{1}{2} \right) e^z}{\cos \pi z} \quad \left| \text{Form } \frac{0}{0} \right| \\ &= \lim_{z \rightarrow \frac{1}{2}} \frac{\left(z - \frac{1}{2} \right) e^z + e^z}{-\pi \sin \pi z} \quad (\text{By L Hospital rule}) \\ &= \frac{e^{1/2}}{-\pi}\end{aligned}$$

Similarly, residue of $f(z)$ at $z = -\frac{1}{2}$ is, $e^{-1/2} / \pi$.

\therefore By residue theorem,

$$\begin{aligned}\int_C \frac{e^z}{\cos \pi z} dz &= 2\pi i \text{ (sum of residues)} \\ &= 2\pi i \left(-\frac{e^{1/2}}{\pi} + \frac{e^{-1/2}}{\pi} \right) = -4i \left(\frac{e^{1/2} - e^{-1/2}}{2} \right) \\ &= -4i \sinh \frac{1}{2}.\end{aligned}$$

Q. 5. (b) Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{x^6 + 1} dx$ by residues.

Ans. Now let,

$$x^3 = X$$

$$3x^2 dx = dX$$

$$x^2 dx = \frac{dX}{3}$$

$$\therefore \frac{1}{3} \int_{-\infty}^{\infty} \frac{dX}{X^2 + 1}$$

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The poles of the $\phi(z) = \frac{1}{z^2 + 1}$ are obtained by solving

$$z^2 + 1 = 0$$

$$z = (-1)^{1/2} = (\cos \pi + i \sin \pi)^{1/2}$$

$$= [\cos(2n\pi + \pi) + i \sin(2n\pi + \pi)]^{1/2}$$

$$z = \left[\cos\left(\frac{2n+1}{2}\pi\right) + i \sin\left(\frac{2n+1}{2}\pi\right) \right] \text{ By De Moivre's theorem}$$

Where $n = 0, 1, 2$

$$n = 0, \quad z = \cos \pi/2 + i \sin \frac{\pi}{2} = i$$

$$n = 1, \quad z = \cos 3\pi/2 + i \sin \frac{3\pi}{2} = -i$$

$$n = 2, \quad z = \cos \frac{5\pi}{2} + i \sin \frac{5\pi}{2} = i$$

Residue of $\phi(z)$ at $ze^{i\pi/2}$ is

$$\begin{aligned} \lim_{z \rightarrow e^{i\pi/2}} \frac{(z - e^{i\pi/2})}{z^2 + 1} &= \lim_{z \rightarrow e^{i\pi/2}} \frac{1}{2z} = \frac{1}{2e^{i\pi/2}} \\ &= \frac{1}{2} e^{-i\pi/2} \end{aligned}$$

$$\text{Residue at } z = e^{3i\pi/2} \text{ is } \frac{1}{2} e^{-3i\pi/2}$$

$$\text{Residue at } z = e^{5i\pi/2} \text{ is } \frac{1}{2} e^{-5i\pi/2}$$

$$\therefore \int_0^{\infty} \frac{x^2}{x^6 + 1} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x^2}{x^6 + 1}$$

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$$= \frac{1}{2} 2\pi i [\text{Sum of residues in the upper half of } s \text{ plane}]$$

$$= \frac{1}{2} 2\pi i \cdot \frac{1}{3} \left[\frac{1}{2} (-i + i) \right]$$

$$= \frac{\pi i}{3} \left(-\frac{1}{2} i \right) = \frac{-\pi}{6} i^2$$

$$= \pi / 6.$$

Part-C

Q. 6. (a) In a bolt factory, machines A, B and C manufacture 25%, 35% and 40% of the total output. Of their output 5%, 4% and 2% are defective bolts. A bolt is drawn at random from the product and is found to be defective. What are the probabilities that it was manufactured by the machines A, B or C?

Ans. Let E_1 , E_2 & E_3 denote the events that a bolt is selected from machine A, B & C respectively.

$$P(E_1) = \frac{25}{100}, P(E_2) = \frac{35}{100}, P(E_3) = \frac{40}{100}$$

$$P(D/E_1) = \frac{5}{100}, P(D/E_2) = \frac{4}{100}, P(D/E_3) = \frac{2}{100}$$

By Baye's theorem,

$$P(D/E_1) = \frac{P(E_1)P(D/E_1)}{\sum_{i=1}^3 P(E_i)P(D/E_i)} = \frac{\frac{25}{100} \times \frac{5}{100}}{\frac{25}{100} \times \frac{5}{100} + \frac{35}{100} \times \frac{4}{100} + \frac{40}{100} \times \frac{2}{100}}$$

$$= \frac{125}{125+140+80} = \frac{125}{345} = \frac{25}{69}$$

$$P\left(\frac{E_2}{D}\right) = \frac{P(E_2)P(D/E_2)}{\sum_{i=1}^3 P(E_i)P(D/E_i)} = \frac{\frac{35}{100} \times \frac{4}{100}}{\frac{125}{(100)^2} + \frac{140}{(100)^2} + \frac{80}{(100)^2}}$$

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$$P\left(\frac{E_3}{D}\right) = \frac{P(E_3)P(D/E_3)}{\sum_{i=1}^3 P(E_i)P(D/E_i)} = \frac{\frac{40}{100} \times \frac{2}{100}}{\frac{25}{(100)^2} \times \frac{5}{(100)^2} + \frac{35}{(100)^2} \times \frac{4}{100} + \frac{40}{100} \times \frac{2}{100}}$$

$$= \frac{80}{125+140+80} = \frac{80}{345} + \frac{16}{69}$$

Q. 6. (b) A function is defined as under :

$$f(x) = \frac{1}{k}, \quad x_1 \leq x \leq x_2$$

$$= 0 \quad \text{otherwise.}$$

Find the cumulative distribution of the variable x when k satisfies the requirement for $f(x)$ to be a density function.

Ans. $f(x) = \frac{1}{k}, \quad x_1 \leq x \leq x_2$

$$= 0, \text{ else where}$$

$\therefore f(x)$ is an density function.

$$\therefore \int_{-\infty}^{\infty} f(x) dx = 1 \quad \int_{x_1}^{x_2} \frac{1}{k} dx = 1$$

$$\Rightarrow \quad k = x_2 - x_1$$

\therefore Cumulative distribution function,

$$F(x) = 0 \text{ if } x < x_1$$

$$= \int_{x_1}^x \frac{1}{k} dx$$

$$= \frac{x - x_1}{x_2 - x_1}; \text{ if } x_1 \leq x \leq x_2$$

$$= 1 \text{ if } x \geq x_2.$$

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Q. 7. (a) In a lot of 500 solenoids 25 are defective, find the probability of 0, 1, 2, 3, defective solenoids in a random sample of 20 solenoids.

Ans. The probability of solenoid to be defective

$$= \frac{25}{500} = \frac{1}{20}$$

The probability of 0 defective

$$= {}^{20}C_0 \left(\frac{1}{20}\right)^0 \left(\frac{19}{20}\right)^{20} = 0.3585$$

The probability of 1 defective

$$= {}^{20}C_1 \left(\frac{1}{20}\right)^1 \left(\frac{19}{20}\right)^{19} = 0.3774$$

The probability of 2 defective

$$= {}^{20}C_2 \left(\frac{1}{20}\right)^2 \left(\frac{19}{20}\right)^{18} = 0.1887$$

The probability of 3 defective

$$= {}^{20}C_3 \left(\frac{1}{20}\right)^3 \left(\frac{19}{20}\right)^{17} = 0.0596$$

The probability of r^{th} defective

$$= {}^{20}C_r \left(\frac{19}{20}\right)^{20-r} \left(\frac{1}{20}\right)^r$$

where $r = 0, 1, 2, 3$.

Q. 7. (b) Fit a Poisson distribution to the set of observations :

x :	0	1	2	3	4
f :	122	60	15	2	1

Ans.

$$\text{Mean} = \frac{\sum f_i x_i}{\sum f_i} = \frac{60 + 36 + 6 + 4}{200} = 0.5$$

\therefore Mean of Poisson distribution i.e., $= 0.5$

Hence the theoretical frequency for r success is

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$$\frac{\lambda_2^{n(r)}(m)}{r!} = \frac{200e^{-0.5}(0.5)^r}{r!} \text{ where } r = 0, 1, 2, 3, 4$$

∴ The theoretical frequencies are,

x:	0	1	2	3	4
f:	121	61	15	2	0

($\because e^{-0.5} = 0.61$)

Q. 8. (a) Using Simplex method :

Maximize $Z = 3x_1 - 2x_2 + 4x_3$

subject to :

$$x_1 + 2x_2 + x_3 \leq 8$$

$$2x_1 - x_2 + x_3 \geq 2$$

$$4x_1 - 2x_2 - 3x_3 = -6;$$

$$x_1, x_2 \geq 0.$$

Ans. **Max.** $Z = 3x_1 + 2x_2 + 4x_3$

Subject to $x_1 + 2x_2 + x_3 \leq 8; 2x_1 - x_2 + x_3 \geq 2$

$$4x_1 - 2x_2 - 3x_3 = -6; x_1, x_2 \geq 0$$

Step 1: The first constraint involves \leq . We introduce only a slack variable s_1 thereby getting

$$x_1 + 2x_2 + x_3 + s_1 = 8$$

The 2nd constraint is greater than type. We introduce a surplus variable s_2 and artificial variable A_1 thereby getting

$$2x_1 - x_2 + x_3 - s_2 + A_1 = 2$$

The 3rd constraint is strict equality and requires neither a slack variable or surplus variable. We add only an artificial variable A_2 these by getting and multiplying by (-1) we get

$$-4x_1 + 2x_2 - 3x_3 + A_2 = 6$$

∴ The standard form of LPP problem becomes,

$$\text{Max., } Z = 3x_1 - 2x_2 + 4x_3 + 0s_1 + 0s_2 + MA_1 + MA_2$$

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$$x_1 + 2x_2 + x_3 + s_1 = 8$$

$$2x_1 - x_2 + x_3 - s_2 + A_1 = 2$$

$$-4x_1 + 2x_2 - 3x_3 + A_2 = 6$$

Where $x_1, x_2, x_3, s_1, s_2, A_1, A_2 \geq 0$ and M is a large positive number.

Step 2 : Since we have 3 equations in 7 variable, a solution is obtained by setting $7 - 3 = 4$ variables equal to zero and solve for remaining variables.

Now an initial basic feasible solution can be obtained by $x_1 = x_2 = x_3 = s_2 = 0$. Therefore the initial basic feasible solution is.

$$s_1 = 8, A_1 = 2, A_2 = 6$$

and
$$Z = 2M + 6M = 8M$$

Therefore the initial basic solution is tabulated as :

Simplex table I :

C_B	Basic	C_j	3	-2	4	0	0	M	N	Ratio
		Solution	x_1	x_2	x_3	s_1	s_2	A_1	A_2	x_B / x_1
		$b (= x_B)$								
0	s_1	8	1	2	1	1	0	0	0	$\frac{8}{1} = 8$
M	A_1	2	2	-1	1	0	-1	1	0	$\frac{2}{2} = 1$
M	A_2	6	-4	2	-3	0	0	0	1	$-\frac{6}{4} = -\frac{3}{2} \rightarrow$
$Z = 8M$		Z_j	-2M	M	-2M	0	-M	M	M	
		$c_j = c_j - z_j$		$3-2M$	$-2-M$	$-2-M \uparrow$	$4+2M$	0	M	0

Some entries in the C_j row being -ve the current solution is not optimal.

Step 3 : Largest negative entry in given is $-2-M$ which lies in x_2 column. Therefore the incoming variable is x_2 . The ratio $-2/3$ is minimum in A_2 row the outgoing basic variable is A_2 . Key element is 2. (In FOR MORE STUDY MATERIAL LOG ON TO <http://studentsuvidha.in>)

further simplex table we will not compute A_2 column) Now key row is

$$2 \quad x_1 \quad 3 \quad -2 \quad 1 \quad -3/2 \quad 0 \quad 0 \quad 0$$

Transformation of R_1 key column entry in R_1 is 2

$$\therefore R_1(\text{new}) = R_1(\text{old}) - 2R_3(\text{new})$$

$$8 - 2(3) = 2$$

$$1 - 2(-2) = 5$$

$$2 - 2(1) = 0$$

$$1 - 2\left(-\frac{3}{2}\right) = 4$$

$$1 - 2(0) = 1$$

$$0 - 2(0) = 0$$

$$0 - 2(0) = 0$$

Transformation of R_2 key column entry in R_2 is (-1)

$$R_2(\text{new}) = R_2(\text{old}) + R_3(\text{new})$$

$$= 2 + 3 = 5$$

$$2 - 2 = 0$$

$$-1 + 1 = 0$$

$$1 - 3/2 = -1/2$$

$$0 + 0 = 0$$

$$-1 + 0 = 0$$

$$1 + 0 = 0$$

New simplex table is given as,

Simplex Table II

C_B	Basis	$C_j \rightarrow$	3	-2	4	0	0	M	Ratio
			x_1	x_2	x_3	s_1	s_2	A_1	x_B / x_3
	solution								

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0	s_1	2	5	0	4	1	0	0	1/2
M	A_1	5	0	0	$-\frac{1}{2}$	0	0	0	$-10 \rightarrow$
2	x_2	3	-2	1	$-\frac{3}{2}$	0	0	0	-2
$Z = 6 + 5M$			Z_j	-4	2	$-3 - \frac{M}{2}$	0	0	0
			\uparrow						

Some entries in C_j row being negative the current solution is not optimal.

Step 4 : Largest negative entry in C_j row $-3 - M/2$ which lies in x_3 -column. Therefore the incoming variable is x_3 . The ratio (-10) is minimum in the A_1 row, therefore the outgoing variable is A_1 . Key constant is $-\frac{1}{2}$. New key row,

$$4 \quad x_3 \quad -16 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0$$

Transformation of R_1 . Key column entry in R_1 is 4

$$\therefore R_{1(\text{new})} = R_{1(\text{old})} - 4R_{2(\text{new})}$$

$$2 - 4(-16) = 66$$

$$5 - 4(0) = 5$$

$$0 - 4(0) = 0$$

$$4 - 4(1) = 0$$

$$1 - 4(0) = 1$$

$$0 - 4(0) = 0$$

$$0 - 4(0) = 0$$

Transformation of R_3 . Key column entry in R_3 is $\left(-\frac{3}{2}\right)$

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$$R_{3(new)} = R_{3(old)} + \frac{3}{2} R_{2(new)}$$

$$3 + \frac{3}{2}(-16) = -21$$

$$-2 + \frac{3}{2}(0) = -2$$

$$1 + \frac{3}{2}(0) = 1$$

$$-\frac{3}{2} + \frac{3}{2}(1) = 0$$

$$0 + \frac{3}{2}(0) = 0$$

$$0 + \frac{3}{2}(0) = 0$$

The new simplex table is given as,

Simplex table III :

C_B	Basic	C_j	3	-2	4	0	0	M	N	Ratio
		Solution	x_1	x_2	x_3	s_1	s_2	A_1	A_2	x_B / x_2
		$b (= x_B)$								
0	s_1	66	5	0	0	1	0	0		
4	x_3	-16	0	0	1	0	0	0		
2	x_2	-21	-2	1	0	0	0	0		
$z = -106$			z_j	-4	2	4	0	0	0	
		$c_j = c_j - z_j$	-1	-4	0	0	0	0		

Continuing in this way we find the optimal solution.

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