B.TECH.

ECS-303

THIRD SEMESTER EXAMINATION 2009-10

DISCRETE MATHEMATICAL STRUCTURE

Time : 3 Hours *Total Marks : 100*

Note: (i) Answer all questions.

(ii) All questions carry equal marks.

Q.1. (a) Let A be set with 10 distinct elements. Determine the following:

(i) Number of different binary relations on set A.

Ans. $^{10}C_2$

(ii) Number of different symmetric binary relations on A.

Ans. 5

(b) Suppose S and T are 2 sets and f is a functions from S to T. Let R1 be equivalence relation on T. Let R2 be a binary relation on S such that $(x,y) \in R2$ if and only if $(f(x), f(y)) \in R1$. Show that R2 is also equivalence relation?

Ans. R2 is equivalence relation on S as it satisfies 3 properties:

1. $(a,a) \in R2$ for all $a \in S$ (reflexive) as $(f(a), f(a)) \in R1$.
2. $(a,b) \in R2$ implies $(b,a) \in R2$ (symmetric) as $(f(a), f(b))$ and $(f(b), f(a)) \in R1$
3. (a,b) and $(b,c) \in R2$ implies $(a,c) \in R2$ (transitive) as $(f(a), f(b))$ and $(f(b), f(c)) \in R1$ implies $(f(a), f(c)) \in R1$.

(c) Prove that union of 2 countably infinite set is countably infinite.

Ans. Consider the sets $A_i = \{ a_{1i}, a_{2i}, a_{3i}, \ldots \}$, $i = 1,2,3,\ldots\ldots$

Each A_i, $i = 1,2,3,4,\ldots\ldots\ldots\ldots$ is countable. The Kth element of A_i is a_{ki}. The elements of the countable union $U A_i$ of the set A_i's can be listed as $a_{11}, a_{12}, a_{13}, a_{14}, a_{23}, a_{32}, a_{41}, \ldots\ldots$ (the order has been taken according to the sum $i + j = k$ for $k = 2, 3, \ldots\ldots$, i, j being the suffices of the element $a_{ij} \in A_j$). The 1-1 correspondence between the elements of $U A_i$ and the set of positive integers is given by $a_{11} \leftrightarrow 1$, $a_{12} \leftrightarrow 2$, $a_{21} \leftrightarrow 3$, $a_{13} \leftrightarrow 4$, $a_{22} \leftrightarrow 5$, $a_{31} \leftrightarrow 6$, $a_{14} \leftrightarrow 7$, $a_{23} \leftrightarrow 8$

(d) Composition of functions is commutative. Prove the statement or give counter example.

Ans. It is not commutative let $f: R \to R$ and $g: R \to R$ and $f(x) = x + 2$ for all $x \in R$ and $g(x) = x^2$ for all $x \in R$

The $(gof)(x) = g(f(x)) = g(x+2) = (x+2)^2 = x^2 + 4x + 4$

And $(fog)(x) = f(g(x)) = f(x^2) = x^2 + 2$

Therefore $gof \neq fog$ since $(gof)(1) = g(f(1)) = g(1+2) = g(3) = 3^2 = 9$ while $(fog)(1) = f(g(1)) = f(1^2) = f(1) = 1 + 2 = 3$

(e) Prove that for any integer n>1

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > \sqrt{n}$$

Ans. $1 + \sqrt{1} + 1 + \sqrt{2} + 1 + \sqrt{3} + \ldots\ldots\ldots + 1 + \sqrt{n} > \sqrt{n}$

1. **Inductive Base:** for n=1, 1/1=1

2. **Inductive Hypothesis :** We assume that the statement is true for $n = k$

i.e $1 + \sqrt{1} + 1 + \sqrt{2} + 1 + \sqrt{3} + \ldots\ldots + 1 + \sqrt{k} > k$

3. **Inductive step:** To prove statement is true for $n = k + 1$

$$K + 1 + \sqrt{k+1} > \sqrt{k+1}$$

Hence proved.

(f) What do you understand by asymptotic behaviour of a numeric function ? Explain Big-oh and Big-Omega notation.

Ans. Asymptotic notation are those which grow as the varable grows.

Big-oh : Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. Then f of order g written as f(x) is O (g(x)) , if there is a constant C and K such that $[f(x)] <= C[g(x)]$ whenever $x > k$

Big-Omega : Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. Then f of order g written as f(x) is O (g(x)) , if there is a constant C and K such that $[f(x)] >= C[g(x)]$ whenever $x > k$

Q.2. Attempt any TWO parts: (10×2=20)

(a) Define group . Prove that if every element of a group G is its own inverse then G is an abelian group.

Ans. Let (G,*) be an algebraic structure where * is a binary operation , then (G,*) is called a group under this operation if the following conditions are satisfied

1. **Closure law:** The binary * is closed operation i.e. a*b ∈ G for all a,b ∈ G
2. **Associative Law:** The binary operation * is an associative operation i.e. a*(b*c) = (a*b)*c for all a,b,c ∈ G
3. **Identity Law :** There exist an identity element i.e. for some e εS , e*a =a*e, a ∈ G.
4. **Inverse Law :** For each a ∈ G , there exist an element a' (the inverse of a) in G such that a* a' = a' * a=e.

Let there be a group G , suc that aa^{-1} belong to G and e be an identity element , e belongs to G then we have the following property

$$aa^{-1} = e \qquad \ldots(i)$$

Now , $(a^{-1})^{-1} \cdot a^{-1} = e = aa^{-1}$ from eq. (i)

$\Rightarrow (a^{-1})^{-1} = a$ from right hand cancellation law.

So the Group is Abelian Group.

(b) (i) Prove the following or give counter example:

If $(H_1, *)$ and $(H_2, *)$ are both subgroups of the group (G, *) then $(H_1 \cap H_2, *)$ is also a subgroup of (G, *)

(ii) State and prove Lagrange theorem.

Ans. Let H be the subgroup of G with order n

Let G is the group of order m

if a ∈ H

Let H_{a1} is the right coset of H

Since G is a finite group therefore no. of right closed with also be finite.

Let it be K

$(G) = (H_{a1}) \cup (H_{a2}) \cup (H_{a3}) \cup (H_{ak})$

$m = n + n + \ldots + k$ times

$m = nk$

$k = m/n$

Proved.

(c) (i) Define and explain the following with suitable example.

    (a) Cyclic group

    (b) Zero divisor of a ring

    (c) Order of an element of a group

    (d) Field

Ans. (a) Cyclic group: A Group G is called cyclic group if for some $a \in G$ every element of G is of the form a^n where n is some integer. The element a is then called generator of G.

(b) Zero Divisor of Ring: If $ab=0$ for all ab element of R where a and b are divisor of zero.

(c) Order of an element of group: n is order of an element of a group G if n is the minimum power of that element s.t.

    $a^n = e$

(d) Field: A ring containing at list 2 elements is called a field if it

(i) is commutative

(ii) has unity

(iii) is such that every non-zero element has multiplicative inverse in R.

(ii) If G is a group of order n then order of any element $a \in G$ is a factor of n. Prove.

(ii) If *G* is a group of order *n* then order of any element *a* ∈ *G* is a factor of *n*. Prove.

Ans. $a \in G$

order of a is n

therefore $a^n = e$.

Clearly a is a factor of n.

Q.3. Attempt any TWO parts: (10×2=20)

(a) (i) Define a relation R on the set Z of all integers by m Rn if and only if $m^2 = n^2$. Determine whether R is a partial order or not?

Ans. m R n

$\Rightarrow$ $m^2 = n^2$ then

Reflexive: mRm $\Rightarrow m^2 = m^2$

Antisymmetric : mRn and nRm $\Rightarrow m^2 = n^2$ or $n^2 = m^2$

Transitive: mRn and nRp $\Rightarrow m^2 = n^2$ and $n^2 = p^2$ $\Rightarrow m^2 = p^2$

(ii) Let (A,<=) and (B,<=) be 2 posets. Prove that (AxB,<=) is a poset, where (a,b) <=(c,d) if and only if a<=c, b<=d

Ans. If $(a,b) \in A \times B$, then (1) $(a,b) \Leftarrow (c,d)$ since $a \Leftarrow c$ in A and $b \Leftarrow d$ in B.

Hence, $\Leftarrow$ is reflexive in AxB

(2) For $(a,b) \Leftarrow (c,d)$ and $(c,d) \Leftarrow (a,b)$ where a and $c \in A$ and b and $d \in B$, we have $a \Leftarrow c$ and $c \Leftarrow a$ in a implies $a = c$

And $b \Leftarrow d$ and $d \Leftarrow b$ implies $b=d$

Since A and B are posets.

Hence <= is antisymmetric in AxB

(3) For $(a,b) \Leftarrow (c,d)$ and $(c,d) \Leftarrow (e,f)$ where a , c,e $\in$ A and b , d,f $\in$ B , we have $a \Leftarrow c$ and $c \Leftarrow e$ in A implies $a \Leftarrow e$

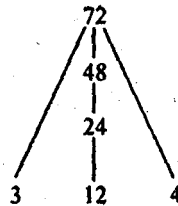And $b \Leftarrow d$ and $d \Leftarrow f$ implies $b \Leftarrow d$

Since A and B are posets.

Hence <= is transitive in AxB

(iii) Draw the Hasse diagram of $(A, \leq)$ where

    A {3, 4, 12, 24, 48, 72}

and the relation $\leq$ be such that $a \leq b$ if a divides b.

Ans.

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          72
         /|
          48
          |
          24
         /|\
        3 12 4
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Q.3.(b) (i) Define distributive lattice and complemented lattice. Prove that in a distributive lattice, if an element has a complement then this complement is unique.

Ans. (i) Distributive Lattice

$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$

$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$

Ans. Complement Lattice

$A \vee A' = A$

$A \wedge A' = 0$

Let a' and a'' be 2 complement then

$A \vee A' = A \quad A \vee A'' = a$

$A \wedge A' = 0 \quad A \wedge A'' = 0$

Therefore a'=a''

(ii) Let $E(x_1, x_2, x_3) = \overline{(x_1 \vee x_2) \vee (\overline{x_1} \wedge x_3)}$ be a boolean expression over the two valued boolean algebra. Write $E(x_1, x_2, x_3)$ in disjunctive normal form.

Ans. $= (x_1 \vee x_2)'' \wedge (x_1' \wedge x_3)'$

$= (x_1 \vee x_2) \wedge (x_1 \vee x_3')$

$= (x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3') \wedge (x_1 \vee x_2' \vee x_3)$

Q.3.(c) Simplify boolean function F given by $F(A,B,C,D) = \Sigma(0, 2, 7, 8, 10, 15)$ using K-map.

Ans. B'D + ABD

Q.4. Attempt any TWO parts: (10×2=20)

Q.4.(a) (i) Given that the value of $p \rightarrow q$ is false, determine the value of $(\overline{p} \vee \overline{q}) \rightarrow q$.

(ii) Find a formula A that uses the variable p, q and r such that A is a contradiction.

(iii) Write an equivalent formula for $p \wedge (q \leftrightarrow r) \vee (r \leftrightarrow p)$ which neither contains biconditional nor conditional connectives.

(iv) The contrapositive of a statement S is given as "If $x < 2$ then $x + 4 < 6$". Write the statement S and its converse.

Ans. (i).

p	Q	p => q	p'	q'	p' ∨ q'	(p'∨q')=>q
T	T	T	F	F	F	T
T	F	F	F	T	T	F
F	T	T	T	F	T	T
F	F	T	T	T	T	F

From the 2nd row we can see that when p=>q is false, (p'vq')=>q is false.

(ii)

P	Q	r	p^q	p^q^r	r'	(p^q^r)^r'
T	T	T	T	T	F	F
T	T	F	T	F	T	F
T	F	T	F	F	F	F
T	F	F	F	F	T	F
F	T	T	F	F	F	F
F	T	F	F	F	T	F
F	F	T	F	F	F	F
F	F	F	F	F	T	F

As all the entries in the last column are false, therefore it is a contradiction.

(iii) p^(q<=>r)v(r<=>p) = p^(((~qvr)^(~rvq))v((~rvp)^(~pvr)))

= p^(((~q^~r)v(q^r))v((~r^~p)v(r^p)))

(iv) Statement : If x + 4 = 6 then x = 2.

Converse : If x = 2 then x + 4 = 6.

Q.4.(b) (i) Prove that $(p \vee q) \Rightarrow (p \wedge q)$ **is logically equivalent to** $p \Leftrightarrow q$.

(ii) Translate the following sentences is quantified expressions of predicate logic.

(a) all students need ifnancial aid.

(b) Some students need financial aid.

Ans. (i)

p	Q	pvq	p^q	(pvq)=>(p^q)	p⇔q
T	T	T	T	T	T
T	F	T	F	F	F
F	T	T	F	F	F
F	F	F	F	T	T

Since all the entries in the last two columns are identical, therefore they are logically equivalent.

(ii) S is a set of all students.

P(x) denote "x need financial help".

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(a) (ix ■ S)P(x)   (Ÿx ■ S)P(x)
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Q.4.(c) (i) Show that following are not equivalent:

(a) $\forall x\ (P(x) \to Q(x))$ and $\forall x\ (P(x) \to \forall x Q(x))$

(b) $\forall x \exists y\ P(x, y)$ and $\exists y\ \forall x\ P(x, y)$

(ii) Show that $r \to \sim q, r \vee s, s \to \sim q, p \to q \Leftrightarrow \sim p$ **are inconsistent.**

Ans. (i) (a) The first one says that for a particular element c, if P(c) then Q(c). But the second one says that for a particular element c, if P(c) then Q(a), Q(b), Q(c), and so on. Therefore , both are not equivalent.

(b) The first one says that for all x, there exits some y, such that, P(x,y). But the second one says that there exists some y for which , for all x, P(x,y).

(ii)

p	q	r	s	r=>~q	rvs	s=>~q	p=>q<=>p
T	T	T	T	F	T	F	T
T	T	T	F	F	T	T	F
T	T	F	T	T	T	F	T
T	T	F	F	T	F	T	F
T	F	T	T	T	T	T	T
T	F	T	F	T	T	T	F
T	F	F	T	T	T	T	T
T	F	F	F	T	F	T	F
F	T	T	T	F	T	F	F
F	T	T	F	F	T	T	F
F	T	F	T	T	T	F	F
F	T	F	F	T	F	T	F
F	F	T	T	T	T	T	F
F	F	T	F	T	T	T	T
F	F	F	T	T	T	T	T
F	F	F	F	F	F	T	T

The last four columns are inconsistent.

Q.5. Answer any TWO parts: $\qquad$ (10×2=20)

5. (a) (*i*) Find the simple expression for the generating function of following discrete numeric function

$$1, \frac{2}{3}, \frac{3}{9}, \frac{4}{27}, \ldots, \frac{(r+1)}{3^r}, \ldots$$

(*ii*) Solve the recurrence relation

$$a_r - 6a_{r-1} + 8a_{r-2} = r \cdot 4^r$$

given $a_0 = 8, a_1 = 22$.

Ans. (i) From the series we see a pattern and conclude that the generating function is of the type,
$\Sigma n/(3^{n-1})$.

(ii) Let a(r)=n^r be a solution of the associated homogeneous recurrence relation

a(r) − 6 · a(r−1) + 8 . a(r−2) = 0

The characteristic equation is

n^2 − 6n + 8 = 0

$n^2 - 4n - 2n + 8 = 0$

$(n - 2)(n - 4) = 0$

$n = 2, 4$

Solution of associated homogeneous recurrence relation is

$a(r) = C1(2)^r + C2(4)^r$

Q.5.(b) (i) Find the number of integer solutions of the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 30$$

where $x_1 \geq 2$, $x_2 \geq 3$, $x_3 \geq 4$, $x_4 \geq 2$, $x_5 \geq 0$

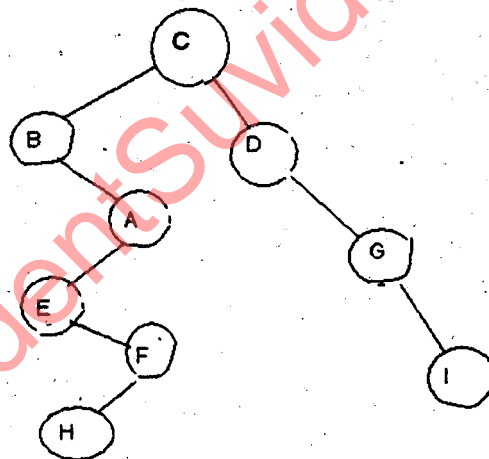**(ii) Given the in order and post order traversal of a tree T**

In order : BEHFACDGI

Post order : HFEABIGDC

Determine the tree T ad its pre order.

(b) (i) Put $x_1 = 2$ a, $x_2 = 3 + b$, $x_3 = 4 + c$, $x_4 = 2 + d$, $x_5 = 0 + e$. The given equation $a + b + c + d + e = 19$ and we seek in non-negative integers a,b,c,d,e. The number of solutions is therefore, $C(19 + 5 - 1, 19) = C(23, 19) = C(23, 4) = 8855$.

(ii)

Pre order : C B A E F H D G I

(c) (i) Prove that for any connected planar graph,

$$V - e + r = 2$$

where v, e, r are the number of vertices, edges and regions of the graph respectively.

Ans. (i) We prove the theorem by induction on e no. of edges.

Basis of induction: If e=0, then graph must have 1 vertex, i.e. v=1 and 1 infinite region, i.e. r=1. Then v-e+r=1-0+1=2. If e=1, then the no. of vertices is either 1 or 2. The first possibility of occurring when edge is a loop. In case of loop v-e+r=1-1+2=2 and in case of non-loop, v-e+r=2-1+1=2. Hence, the result is true.

Induction Hypothesis: Suppose that the result is true for any connected graph G with e-1 edges.

Induction Step: We add 1 new edge k to G to form a connected super graph of G, which is denoted by G+k. There are following 3 cases:

(i) k is a loop, so vertices remains unchanged.

(ii) k joins 2 vertices of G, so that no. of regions is increased by 1.

(iii) k is incident with only 1 vertex, so another vertex must be added.

If let v',e' and r' denote the vertices, edges and regions in G and v,e and r denote the same in G+k, then in case

(i) $v - e + r = v' - (e' + 1) + (r' + 1) = v' - e' + r'$

(ii) $v - e + r = v' - (e' + 1) + (r' + 1) = v' - e' + r'$

(iii) $v - e + r = (v' + 1) - (e' + 1) + r' = v' - e' + r'$

But by our induction hypothesis, $v' - e' + r' = 2'$ hence in each case, $v - e + r = 2$. Hence, proved.

(ii) Define and explain the following:

(a) Bipartite graph\

(b) Chromatic number of a graph

(c) Binary search tree

(d) Adjacency matrix of a graph.

Ans. (a) Bipartite Graph: A graph G = (V, E) is bipartite if vertex at V can be partitioned into 2 subsets V_1 and V_2 such that every edge in E connects a vertex in V_1 and a vertex V_2 so that no edge in G connects either 2 vertices in V_1 or V_2.

(b) Chromatic Number: Chromatic number of a graph G is the minimum number of colors to colour the vertices of graph G and is denoted by $\chi(G)$. A graph G is n-colorable if $\chi(G) <= n$.

(c) Binary Search Tree : A binary search tree is a binary tree in which the data are associated with the vertices. The data are arranged so that, for each vertex V, each data item in the left sub tree of V is less than the data item in V and each data item in the right sub tree of V is greater than the data item in V.

(d) Adjacency Matrix: The adjacency matrix of a graph G with n vertices is a n*n matrix $A = \{a_{ij}\}$, whose elements are given by,

$A_{ij} = 1$, if there is path between the two vertices

$A_{ij} = 0$, if there is no path.